How to analyze the general two-Higgs-doublet model

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Workshop on Multi-Higgs Models, Lisbon, September 16-18, 2009

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Why to study general 2HDM?

Two-Higgs-doublet model suggested by T.D.Lee in 1973:

$$\phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}.$$

Constructing the model:

- the scalar and Yukawa sectors has dozens of free parameters; some of them are restricted by experiment.
- interesting phenomenology in the scalar and fermion sectors appears even with very few non-zero free parameters.
- $\rightarrow$ many specific variants of 2HDM have been developed in the past decades.

Why to study general 2HDM? (cont.)

Instead of specific variants of 2HDM, I will discuss scalar sector of the general 2HDM, i.e. the Higgs potential with all possible EW-invariant quadratic and quartic combinations of ϕ_1 and ϕ_2.

Why does the general case require a special treatment?
Because of an "algebraic barrier": explicit minimization of the potential is impossible.

I will show a geometric approach to 2HDM, which allows one to bypass the algebraic barrier and get at least some insight into the general 2HDM without the need to explicitly minimize the potential.

Why to study general 2HDM? (cont.)

For phenomenology, one needs to looks simultaneously at the Higgses and at the fermions. However, recently the properties of the Higgs potential alone (even at the tree level!) have attracted much attention.

- logically, one first minimizes the potential and only then calculates the masses and interactions of physical particles.
- sometimes different sets of parameters of the potential lead to similar phenomenology $\rightarrow$ idea of hidden symmetry.
- $\rightarrow$ The "space of 2HDM models" must have some structure: some parameters of the potential are essential, some are redundant. How to know which are which?
- What symmetries are in principle possible in the scalar sector of 2HDM? How are they related with the parameters of the potential? Which of them can be extended to the fermion sector?

Why to study general 2HDM? (cont.)

The notions of hidden symmetry and equivalence between models is made precise with the technique of reparametrization transformation.

- Reparametrization transformations are transformations of the Higgs fields, which leave the general form of the Higgs potential unchanged, but just induce some transformation of the parameters.
- If two models with different sets of parameters are related by a certain reparametrization transformation, they lead to the same physics.
- Only reparametrization-invariant combinations of the parameters are essential; all the other are redundant.

But even if we start from a very restricted 2HDM, reparametrization transformations will lead us to the most general 2HDM.

Why to study general 2HDM? (cont.)

What could we learn, if we had a complete description of the scalar sector in the general 2HDM?

- One would get the entire spectrum of possibilities offered by the second doublet.
- Relations among particular models the specific models should become clearer.
- Construction of models with predefined symmetries $\rightarrow$ a useful experience for even more complicated Higgs sectors.
- One would understand how stable are the results obtained in specific models.

General 2HDM should be viewed as a useful tool rather than an attempt to accurately describe Nature.

STEP ONE:

Finding structure behind 2HDM

Structure behind 2HDM

The most general Higgs potential: $V = V_2 + V_4$

$$V_2 = -\frac{1}{2}\left[m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{12}^2(\phi_1^\dagger\phi_2) + m_{12}^{2\,*}(\phi_2^\dagger\phi_1)\right] ;$$

$$\begin{aligned} V_4 = &\frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\ &+\frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + \lambda_5^*(\phi_2^\dagger\phi_1)^2\right] \\ &+\left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \mathrm{h.c.}\right\} \end{aligned}$$

$4 + 10 = 14$ free parameters

The main problem: it cannot be minimized explicitly (coupled algebraic equations of high order);

Structure behind 2HDM (cont.)

Think of ϕ_1 and ϕ_2 as components of $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.

The key property of the generic potential: if we perform any linear transformation of Φ,

$$\Phi \to \Phi' = A \cdot \Phi \,, \quad A \in GL(2, C)\,,$$

we still obtain the generic potential, but with reparametrized coefficients m_{ij}^2 and λ_i.

It is always possible to perform simultaneous transformations of ϕ_i and of coefficients so that the potential does not change at all $\to$ we have reparametrization freedom with the reparametrization group $GL(2, C)$.

$GL(2, C) =$ overall multiplication $\times$ $SL(2, C)$.

Structure behind 2HDM (cont.)

Potential depends on $(\phi_i^\dagger \phi_j)$, $i,j = 1,2$ (EW orbits). Let's organize them into combinations:

$$r_0 = (\Phi^\dagger \Phi) \equiv (\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2)\,,$$
$$r_i = (\Phi^\dagger \sigma^i \Phi) \equiv \left(\begin{array}{c} 2\mathrm{Re}\,(\phi_1^\dagger \phi_2) \\ 2\mathrm{Im}\,(\phi_1^\dagger \phi_2) \\ (\phi_1^\dagger \phi_1) - (\phi_2^\dagger \phi_2) \end{array} \right).$$

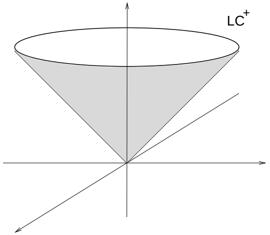
When Φ is transformed by $A \in SL(2,C)$, r_0 and r_i transform as a single 4-vector $r^\mu = (r_0,\ r_i)$. The reparametrization group in the orbit space is the proper Lorentz group $SO(1,3)$.

Structure behind 2HDM (cont.)

The shape of the orbit space

$$r_\mu r^\mu = 4\left[(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - (\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)\right] \geq 0$$
$$r_0 = (\phi_1^\dagger\phi_1) + (\phi_2^\dagger\phi_2) \geq 0$$

The orbit space is the surface and interior of the "future light-cone" (LC^+) in the Minkowski space.



Structure behind 2HDM (cont.)

The Higgs potential is just quadratic form in the orbit space:

$$V = -M_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu} r^\mu r^\nu \,.$$

The kinetic term in the lagrangian must be also treated in the reparametrization invariant way:

$$K = K_\mu \rho^\mu \,, \quad \rho^\mu \equiv (D_\alpha \Phi)^\dagger \sigma^\mu (D^\alpha \Phi)\,.$$

the "standard" K_μ being (1, 0, 0, 0).

All properties of the most general 2HDM come from the relative "orientation" of $\Lambda_{\mu\nu}$, M_μ, K_μ.

Structure behind 2HDM (cont.)

$$M_\mu = \frac{1}{4}\left(m_{11}^2 + m_{22}^2,\ -2\mathrm{Re}\ m_{12}^2,\ 2\mathrm{Im}\ m_{12}^2,\ -m_{11}^2 + m_{22}^2\right)\ ,$$

$$\Lambda_{\mu\nu} = \frac{1}{2}\begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} + \lambda_3 & -\mathrm{Re}\ (\lambda_6+\lambda_7) & \mathrm{Im}\ (\lambda_6+\lambda_7) & -\frac{\lambda_1-\lambda_2}{2} \\ -\mathrm{Re}\ (\lambda_6+\lambda_7) & \lambda_4 + \mathrm{Re}\ \lambda_5 & -\mathrm{Im}\ \lambda_5 & \mathrm{Re}\ (\lambda_6-\lambda_7) \\ \mathrm{Im}\ (\lambda_6+\lambda_7) & -\mathrm{Im}\ \lambda_5 & \lambda_4 - \mathrm{Re}\ \lambda_5 & -\mathrm{Im}\ (\lambda_6-\lambda_7) \\ -\frac{\lambda_1-\lambda_2}{2} & \mathrm{Re}\ (\lambda_6-\lambda_7) & -\mathrm{Im}\ (\lambda_6-\lambda_7) & \frac{\lambda_1+\lambda_2}{2} - \lambda_3 \end{pmatrix}.$$

Structure behind 2HDM (cont.)

Positivity constraint

$V_4 > 0$ for all non-zero ϕ_i
$\rightarrow \Lambda_{\mu\nu} r^\mu r^\nu > 0$ on and inside LC^+. This holds, iff $\Lambda_{\mu\nu}$ can be diagonalized by an $SO(1,3)$ transformation, and after diagonalization $\Lambda_{\mu\nu}$ takes form

$$\begin{pmatrix} \Lambda_0 & 0 & 0 & 0 \\ 0 & -\Lambda_1 & 0 & 0 \\ 0 & 0 & -\Lambda_2 & 0 \\ 0 & 0 & 0 & -\Lambda_3 \end{pmatrix} \quad \text{with} \quad \Lambda_0 > 0 \text{ and } \Lambda_0 > \Lambda_1, \Lambda_2, \Lambda_3 \,.$$

One can establish this without any need to know the explicit expressions of Λ_i in terms of original λ's of the potential!

STEP TWO:

Geometric analysis of extrema

Search for the extrema

Three kinds of extrema:

- EW-symmetric vacuum: $\langle\phi_i\rangle = 0 \;\to\; \langle r^\mu\rangle = 0$.
- Charge-breaking vacuum: $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ are not proportional:

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ v_2 \end{pmatrix}$$

  with $u \neq 0$. Then $\langle r^2\rangle > 0$: the interior of the lightcone. The extremum is unique; conditions when it is the global minimum are known.

- Neutral vacuum: doublets $\langle\phi_1\rangle$ and $\langle\phi_2\rangle$ are proportional to each other ($u = 0$) $\to$ $\langle r^2\rangle = 0$: the surface of the lightcone. Up to six extrema (minima or saddle points).

The question is to find how many among these six are minima and what are their properties.

Telling a minimum from a saddle point

Physically realizable r^μ lies on and inside LC^+. But consider V in the entire Minkowski space. If $\Lambda_{\mu\nu}$ is non-singular, then

$$V = -M_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu} r^\mu r^\nu = \frac{1}{2}\Lambda_{\mu\nu}(r^\mu - m^\mu)(r^\nu - m^\nu) + V_0\,,$$

where $m_\mu = (\Lambda^{-1})_{\mu\nu} M^\nu$.

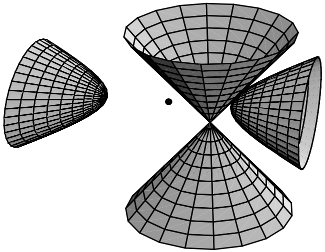
An equipotential surface is simply a 3-quadric (3-hyperboloid, 3-ellipsoid) $\Lambda_{\mu\nu} p^\mu p^\nu$ constructed from $p^\mu = r^\mu - m^\mu$.

The family of equipotential surfaces is just the family of nested 3-quadrics.

Telling a minimum from a saddle point (cont.)

The problem of minimization is reduces to study of intersections of the equipotential surfaces with LC^+.

The unique equipotential surface that only touches but does not intersect LC^+ gives the lowest possible value of V realizable in the physical Higgs space. Their contact points are the positions of the global minimum $\langle r^\mu \rangle \to \langle \phi_1 \rangle, \langle \phi_2 \rangle$.



Telling a minimum from a saddle point (cont.)

- Since we study contact of second-order manifolds, the global minimum cannot be degenerate more than twice.
- Less trivial result: There cannot be more than two distinct minima at all.

NB: existence of a second minimum is very often overlooked, especially when one parametrizes the Higgs potential starting from a minimum. One must always check that the so constructed potential does not possess a deeper minimum!

Symmetries and their violation

Geometric approach offers a novel look at the symmetries of 2HDM.

- It is usually believed that symmetries are very atypical for a general 2HDM,
- However, one can show that the quartic part of the potential

$$
\begin{aligned}
V_4 = & \frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\
& + \frac{1}{2}\left[\lambda_5(\phi_1^\dagger\phi_2)^2 + \lambda_5^*(\phi_2^\dagger\phi_1)^2\right] \\
& + \left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \mathrm{h.c.}\right\}
\end{aligned}
$$

subject to positivity constraints always has at least a $(Z_2)^3$-symmetry. It is highly non-trivial if you look at this form of V_4, but it becomes obvious in the geometric approach.

Symmetries and their violation (cont.)

- Similarly, the quadratic part of the potential always has at least an $O(2)$-symmetry. The same is valid for the kinetic term.
- The symmetry group of the model is the intersection of these three symmetry groups: the Higgs lagrangian has an additional explicit symmetry, iff $\Lambda_{\mu\nu} r^\mu r^\nu$, $M_\mu r^\mu$, $K_\mu \rho^\mu$ are invariant under some transformation of the Minkowski space.
- Another insight: the tree-level potential can have symmetries not shared by the kinetic term. They play a special role in the analysis of the 2HDM and deserve a closer study.

Symmetries and their violation (cont.)

All possible explicit symmetries as well as their spontaneous violation can be studied within the above geometric approach.

Classes of possible explicit symmetries (in the orbit space): Z_2, $(Z_2)^2$, $(Z_2)^3$, $O(2)$, $O(2) \times Z_2$, $O(3)$.

Criterion for the existence of an explicit symmetry:

> There exists an eigenvector of $\Lambda_{\mu\nu}$ which is orthogonal to both M_μ and K_μ.

Symmetries and their violation (cont.)

Explicit CP-conservation means that the Higgs lagrangian is invariant under reflection of the second axis (the one coupled to $Im(\phi_1^\dagger \phi_2)$): in the $\Lambda_{\mu\nu}$-diagonal frame, $M_2 = 0$, $K_2 = 0$.

Necessary and sufficient conditions for spontaneous CP-violation:

- existence of extremum: M^μ lies inside the ellipse:

$$\frac{M_1^2}{(\Lambda_1 - \Lambda_2)^2} + \frac{M_3^2}{(\Lambda_3 - \Lambda_2)^2} < \frac{M_0^2}{(\Lambda_0 - \Lambda_2)^2}\,.$$

- the extremum is minimum: $\Lambda_2 > \Lambda_1, \Lambda_3$.

Symmetries and their violation (cont.)

Some further theorems:

- For any explicit discrete symmetry, the symmetry-conserving and symmetry-violating minima cannot coexist.
- Any explicit discrete symmetry is given by the group $(Z_2)^k$, $k = 1, 2, 3$. The maximal spontaneous violation consists in removing just one Z_2 factor (= too symmetric 2HDM cannot have spontaneous CP-violation).

Masses of the general 2HDM

- Masses of the physical Higgs bosons are reparametrization-invariant quantities $\to$ they must be expressible in terms of $\Lambda_{\mu\nu}$, M_μ, K_μ.
- However, electroweak indices "open up", when one differentiates the potential. Intermediate calculations must be conducted in terms of fields, not bilinears r^μ.
- The mass-matrix $\mathcal{M}$ is reparametrization-dependent, but its eigenvalues (the masses) are not. Calculate $\mathcal{M}$ in a specific basis, find $\mathrm{Tr}(\mathcal{M}^n)$, rewrite them in rep.-covariant way.
- Non-diagonal kinetic term must be taken into account.

Masses of the general 2HDM (cont.)

Sketch of the procedure:

- Switch from two complex doublets ϕ_i to 8 real fields φ_a:

$$r^\mu = \phi_i^\dagger \sigma^\mu_{ij} \phi_j \equiv \varphi_a \Sigma^\mu_{ab} \varphi_b \,.$$

  8×8 real symmetric matrices Σ^μ_{ab} share some of the properties of σ^μ.

- Calculate the hessians of the kinetic term and potential:

$$K_{ab} = \frac{\partial^2 K}{\partial(\partial_\mu \varphi_a)\,\partial(\partial^\mu \varphi_b)}\,, \quad H_{ab} = \frac{\partial^2 V}{\partial \varphi_a\,\partial \varphi_b}\,.$$

- The mass matrix is $\mathcal{M} = K^{-1} H$.
- One can calculate the traces and express them in a covariant way.

Masses of the general 2HDM (cont.)

Example: charge-breaking vacuum.

$$K_{ab} = K_\mu \Sigma^\mu_{ab}\,; \quad H_{ab} = 2\Lambda_{\mu\nu}\Sigma^\mu_{aa'}\langle\varphi_{a'}\rangle\langle\varphi_{b'}\rangle\Sigma^\nu_{b'b}\,,$$

where $\langle\varphi_a\rangle$ indicates fields at the extremum:

$$\langle\varphi_a\rangle\Sigma^\mu_{ab}\langle\varphi_b\rangle = (\Lambda^{-1})^{\mu\nu}M_\nu\,.$$

Trace of the mass matrix:

$$Tr(\mathcal{M}) = 4K_\mu M^\mu - 2\,Tr\Lambda\,K_\mu(\Lambda^{-1})_{\mu\nu}M^\nu\,.$$

Traces of powers of $\mathcal{M}$ are also calculable → properties of its eigenvalues can be inferred.
Mass matrices for the neutral vacua are analyzed in a similar way.

Conclusions

usual parameters

$\lambda_i, \ m_{ij}^2$

$\not\Downarrow$

phenomenology:

v_i, M_{Hi}, symmetries

Cross $\times$ indicates an "algebraic barrier".

Conclusions (cont.)

usual parameters		$\Lambda_{\mu\nu}$-diagonal frame:
$\lambda_i,\ m_{ij}^2$	$\not\longrightarrow$	$\Lambda_i,\ M_\mu,\ K_\mu$
$\not\downarrow$		$\downarrow$
phenomenology:		general results:
v_i, M_{Hi}, symmetries	$\not\longleftarrow$	minima, coexistence, symmetries

Cross $\times$ indicates an "algebraic barrier".

Conclusions (cont.)

Conclusions:

- The Higgs sector of the most general 2HDM can be studied without explicitly solving the high-order algebraic equations. The key step is the observation that the space of all 2HDM models has the Minkowski space structure.
- The number, the properties and the coexistence of the minima of the Higgs potential can be studied in geometric terms $\to$ phase diagram of the general 2HDM can be reconstructed.
- Higgs mass spectrum of the general 2HDM can be analyzed in a reparametrization-invariant way.

Extra slides

Many of these questions can be answered in the geometric approach, at least, in the tree-level approximation:

- How many minima can the 2HDM potential have?
- When is the global minimum degenerate? How is it related to the symmetries of the model? What are the possible symmetries of 2HDM and when they are spontaneously broken?
- What is the phase diagram of the model and what phase transitions can take place upon continuous change of parameters?

Extra slides (cont.)

Another insight: the tree-level potential can have symmetries, not shared by the kinetic term.

Such symmetries play a special role:

- Degeneracy of two local minima automatically implies existence of a discrete symmetry of the potential, but not necessarily of the entire lagrangian.
- Possible first-order thermal phase transitions are associated with a momentary restoration of a discrete symmetry of the potential.

Such symmetries deserve a closer study.

Extra slides (cont.)

Three kinds of extrema depending on position of $\langle r^\mu \rangle$:

- $\langle r^2 \rangle > 0$ if and only if doublets ϕ_1 and ϕ_2 are not proportional.

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ v_2 \end{pmatrix}$$

  with $u \neq 0$. Photon is coupled to the Higgs field $\rightarrow$ is becomes massive $\rightarrow$ charge-violating vacuum (might have taken place in the early Universe).

- $\langle r^2 \rangle = 0$ if and only if doublets ϕ_1 and ϕ_2 are proportional. Then $u = 0$, and photon is massless $\rightarrow$ neutral vacuum
- $\langle r^\mu \rangle = 0 \rightarrow$ EW-symmetric vacuum.

Extra slides (cont.)

Two situations:

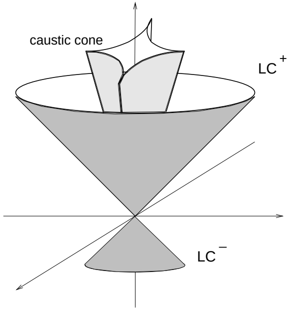
- $\langle r^2 \rangle > 0$: $\partial V/\partial r^\mu = -M_\mu + \Lambda_{\mu\nu}\langle r^\nu \rangle = 0$.
  For a non-degenerate $\Lambda_{\mu\nu}$ a solution always exists, but it is physically realizable only if $\langle r^\mu \rangle$ lies inside LC^+. It is a minimum if and only if $\Lambda_{\mu\nu}$ is positive definite in the entire Minkowski space. (= all $\Lambda_i < 0$).
- $r^2 = 0$: Lagrange multipliers method,
  $-M_\mu + \Lambda_{\mu\nu}\langle r^\nu \rangle - \lambda \langle r_\mu \rangle = 0$.
  Up to six solutions (minima or saddle points) are possible, depending on position of M_μ.

  NB: negative Lagrange multiplier, $\lambda < 0$, automatically corresponds to saddle points.

Extra slides (cont.)

Potential stable in a strong sense:

- at least one neutral extremum exists if M^μ lies outside LC^-,
- if M^μ lies inside LC^+, at least two neutral extrema exist;
- if M^μ, in addition, lies inside caustic cones, two additional extrema (per cone) appear.



Potential stable in a weak sense: if M^μ lies outside LC^-, there is only one non-trivial extremum (= global minimum).
Phase diagram of 2HDM at the tree-level.

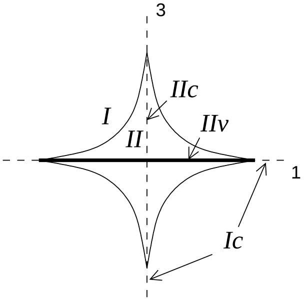
Example: all coefficients and Higgs fields are real → second axis can be dropped → $1+2$-Minkowski space.

Extra slides (cont.)

M_1, M_3-slice of the phase diagram at $\Lambda_3 > \Lambda_1$, $K_i = 0$.

I: 1 minimum, no symmetry;
II: 2 minima, no symmetry;
Ic: 1 minimum, symmetry conserved;
IIc: 2 minima, symmetry conserved;
IIv: 2 minima, symmetry violated;

thick line: first-order phase transition;
end points: critical points.



Ginzburg-Landau model with two order parameters

The same approach can be used to study condensed matter problems that can be formulated via Ginzburg-Landau model with two order parameters (two-band superconductors, e.g. MgB_2; charge density waves, non-conventional superfluidity etc.)

Initial analysis (I.P.I., PRE 79, 021116 (2009)) includes:

- Extrema/minima/symmetries of the Landau potential;
- Complete description of the phase diagram;
- Complete description of the surfaces/lines/points of the 1st or 2nd-order phase transitions;
- Calculation of some critical exponents via geometry;
- Origin of metastable topologically non-trivial configurations.