Symmetries in the three-Higgs-doublet model

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Workshop on Multi-Higgs Models, Lisbon, August 28-31, 2012

in collaboration with Venus Keus (Liege) and Evgeny Vdovin (Novosibisk);
based on J. Phys. A45, 215201 (2012), on arXiv:1206.7108, and on work in progress

What this talk is about

I am not going to:

- promote any specific bSM model,
- or give detailed predictions for the LHC or astroparticle observables.

I will present some general results on what's possible, symmetry-wise, in models with N Higgs doublets (NHDM).

Our motivaton is very pragmatic: many people study particular variants of NHDM based on various symmetry groups. Which group to pick is often a matter of one's taste; no complete list of "allowed" groups is known.

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Multi-Higgs-doublet models

NHDMs are among the most actively studied bSM models of EWSB:

- Conceptually simple: "Higgs generations".
- 2HDM is used in MSSM and is interesting on its own.
- Many specific variants of NHDM for $N \geq 3$ were studied (one-paper-per-group list):

Weinberg, PRL37, 657 (1976); Adler, PRD59, 015012 (1999);
Ma, Rajasekaran, PRD64, 113012 (2001); Ferreira, Silva, PRD78, 116007 (2008);
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Multi-Higgs-doublet models

One particularly interesting question concerns symmetries (in addition to the gauge group) which can be implemented in the scalar sector of NHDM.

These additional symmetries have an impact on phenomenological and astroparticle aspects of the model, so it is important to know which symmetry groups can arise with N doublets.

Although several particular symmetry groups have been implemented and studied, the full classification is still missing for $N > 2$.

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The scalar sector of NHDM

We introduce ϕ_a, $a = 1, \dots, N$, and construct the general gauge-invariant and renormalizable potential from $(\phi_a^\dagger \phi_b)$'s:

$$V = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{abcd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,,$$

with N^2 independent components in Y and $N^2(N^2+1)/2$ independent components in Z (e.g. 14 free parameters for 2HDM, 54 free parameters for 3HDM).

Reparametrization transformation: any transformation of the doublets which keeps the generic form of the potentials but only change the values of free parameters.

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Technical point 1: $PSU(N)$

Here we focus only on Higgs-family transformations: unitary transformations in the space of N doublets.

A priori, these transformations form the group $U(N)$.

$U(N)$ contains the subgroup of overall phase rotations, which is already included in the gauge group $U(1)_Y$. But we want to study structural symmetries of the NHDM potentials, so we should disregard transformations which leave all the EW-invariant potentals by constructions.

This leads us to the group $U(N)/U(1) \simeq SU(N)$.

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However, there still remain overall phase change rotations inside $SU(N)$: $\mathrm{diag}(e^{2\pi i/N} \dots, e^{2\pi i/N})$. They form the center of the group, $Z(SU(N)) \simeq \mathbb{Z}_N$.

Again, they act trivially on all EW-invariant potentials. Therefore, if we want to study structural properties of NHDM, we need to consider the factor group

$$SU(N)/Z(SU(N)) = PSU(N)\,.$$

All reparametrization symmetry groups we describe below are subgroups of $PSU(3)$, not $SU(3)$.

Technical point 2: Realizable symmetry groups

Definition:
we call a symmetry group G realizable if there exists a G-symmetric potential which is not symmetric under a larger group containing G.

Good points about realizable groups:

- it represents the full symmetry content of the corresponding potential;
- the symmetry group of the vacuum is guaranteed to be a subgroup of the symmetry group of the potential.

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Some examples when the condition is not satisfied:

- *Ferreira, Silva, PRD 78, 116007 (2008)* described situations in 2HDM and 3HDM when imposing a discrete group of rephasing transformations led to potentials with continuous symmetry groups (which is a dangerous situation due to possible goldstone modes);
- *de Adelhart Toorop et al, JHEP 1103, 035 (2011)* studied the A_4-symmetric 3HDM and found that at certain values of the parameters the symmetry group of the vacuum was $S_3 \not\subset A_4$. This happened because at these parameters the true symmetry group of the potential is in fact S_4.

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Symmetries in NHDM

In 2HDM, all questions regarding symmetries have been answered (see e.g. review *Branco et al, Phys. Rept. 516, 1 (2012)*). The full list of symmetry groups realizable in 2HDM is

$$\mathbb{Z}_2,\ (\mathbb{Z}_2)^2,\ (\mathbb{Z}_2)^3,\ U(1),\ U(1)\times\mathbb{Z}_2,\ SU(2)\,.$$

For any $N > 2$, despite several attempts, the classification was still missing.

We solved this problem for 3HDM. I will present here the key part of our analysis: derivation of the list of finite Higgs-family symmetry groups realizable in 3HDM.

Antiunitary (i.e. generalized-CP transformations) have also been included; including continuous symmetries is rather straightforward.

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Outline of the strategy

Three steps towards classification of finite symmetry groups in 3HDM:

- Since abelian groups are basic building blocks of any group, find first all relevant finite abelian symmetry groups in 3HDM.
- Group-theoretic part: prove that any finite symmetry group G must satisfy $G/A \subseteq Aut(A)$, where A is one of the abelian groups found previously. So, G can be constructed from A by extension.
- Calculational part: check all possible A's and extensions and see whether the potential supports this group.

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Finite abelian symmetry groups in 3HDM

Realizable abelian symmetry groups in NHDM for any N were completely characterized in *Ivanov, Keus, Vdovin, J. Phys. A45, 215201 (2012)*.

For 3HDM, the following finite groups are relevant for your study:

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_3 \times \mathbb{Z}_3 .$$

This list is complete: imposing any other finite abelian symmetry group on the 3HDM scalar potential unavoidably leads to continuous symmetry group.

Note that the orders of these groups, $|A|$, have only two prime factors: 2 and 3.

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Group-theoretic part

- The finite symmetry group G must contain only these abelian subgroups
- $\Rightarrow$ the order of the group $|G| = 2^a 3^b$
- $\Rightarrow$ by Burnside's $p^a q^b$ theorem, G is solvable
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Group-theoretic part

So far, we don't have any restriction on the size of G/A.

However we proved that this A can be chosen to be a maximal normal abelian subgroup of G. "Maximal abelian" means that it is not contained in a larger abelian subgroup of G. That is, there is no element of $G \setminus A$ can commute with the entire A:

$$\text{the situation} \quad g^{-1}ag = a \quad \forall a \in A \quad \text{is impossible.}$$

Then, all non-unit elements of $G \setminus A$, acting by conjugation $A \to g^{-1}Ag$, must induce non-trivial automorphisms on A.

$$G/A \subseteq Aut(A)\,,$$

and G can then be constructed as an extension of A by $K \subseteq Aut(A)$.

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Constructing G by extensions: $\mathbb{Z}_4$ example

Example: $A = \mathbb{Z}_4$. Then $Aut(\mathbb{Z}_4) = \mathbb{Z}_2$, so G is extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$.

There are several possibilities.

(1) extensions which lead to larger abelian groups ($\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$) are immediately excluded;

(2) dihedral group D_8, the symmetry group of the square.

$$D_8 = \langle a, b \rangle \text{ with conditions } a^4 = 1,\ b^2 = 1,\ ab = ba^3 .$$

If $a = \mathrm{diag}(i, -i, 1)$, then

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{with arbitrary } \delta.$$

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A generic $\mathbb{Z}_4$ potential can be brought to the form $V_0 + V_{\mathbb{Z}_4}$, where

$$V_0 = -\sum_a m_a^2(\phi_a^\dagger\phi_a) + \sum_{a,b}\lambda_{ab}(\phi_a^\dagger\phi_a)(\phi_b^\dagger\phi_b) + \sum_{a\neq b}\lambda'_{ab}(\phi_a^\dagger\phi_b)(\phi_b^\dagger\phi_a)\,,$$

and

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2) + \lambda_2(\phi_1^\dagger\phi_2)^2 + h.c.$$

The λ_1 term is invariant under any b, while the λ_2 term transforms as

$$(\phi_1^\dagger\phi_2)^2 \mapsto e^{-4i\delta}(\phi_2^\dagger\phi_1)^2\,.$$

If we restrict parameters of V_0 ($m_{11}^2 = m_{22}^2$, $\lambda_{11} = \lambda_{22}$, $\lambda_{13} = \lambda_{23}$, $\lambda'_{13} = \lambda'_{23}$) then the potential is symmetric under one particular D_8 group in which the value of $\delta = \arg\lambda_2/2$.

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How to prove that the potential has no other Higgs-family symmetry beyond this D_8?

- absence of continuous symmetries is related to the fact that we don't have terms like $(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_3)$.
- absence of higher discrete symmetry follows from the fact that we know conditions for all other finite groups. A generic D_8 potential does not satisfy them.

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Constructing G by extensions: $\mathbb{Z}_4$ example

(3) quaternion group Q_8:

$$Q_8 = \langle a, b \rangle \text{ with conditions } a^4 = 1,\ b^2 = a^2,\ ab = ba^3\,.$$

If $a = \mathrm{diag}(i, -i, 1)$, then

$$b(Q_8) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Again, the $\mathbb{Z}_4$ part of the potential:

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2) + \lambda_2(\phi_1^\dagger\phi_2)^2 + h.c.$$

Upon this b, the λ_1 term changes its sign. The only way to impose Q_8 is to set $\lambda_1 = 0$. But then the potential becomes invariant under a continuous transformation: $\mathrm{diag}(e^{i\alpha}, e^{i\alpha}, 1)$.

We conclude that Q_8 is not realizable in 3HDM.

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Finite Higgs-family symmetry groups in 3HDM

We performed this kind of analysis for all abelian groups we have.

Results:

$$\mathbb{Z}_2\,,\quad \mathbb{Z}_3\,,\quad \mathbb{Z}_4\,,\quad \mathbb{Z}_2\times\mathbb{Z}_2\,,$$
$$D_6\simeq S_3\,,\quad D_8\,,\quad T\simeq A_4\,,\quad O\simeq S_4\,,$$
$$(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2=\Delta(54)/\mathbb{Z}_3\,,\quad (\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_4=\Sigma(36)\,.$$

This list is complete: trying to impose any other finite Higgs-family symmetry group on the 3HDM potential will lead to a potential symmetric under a continuous group.

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Possible uses

- Patterns of spontaneous breaking of each of these symmetry groups can be studied → first systematic analysis of scalar sector phenomenologies possible with three doublets [*work in progress*].
- Examples of scalar dark matter models based on group $\mathbb{Z}_p$ rather than $\mathbb{Z}_2$ with desired microscopic dynamics can be easily constructed: talk by Venus Keus this afternoon.
- The symmetry patterns the scalars generate in the Yukawa sector can be investigated (along the same lines as shown in the talk by Heinrich Päs).

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