3HDM w/ $\Delta(27)$ from an outer automorphism perspective

Andreas Trautner

based on

NPB883 (2014) 267-305 (arXiv:1402.0507)
with: M.–C. Chen, M. Fallbacher, K.T. Mahanthappa and M. Ratz.

NPB894 (2015) 136-160 (arXiv:1502.01829)
with: M. Fallbacher.

CFTP
6.9.16

Motivation

- Standard Model flavor puzzle.
- Flavor and CP are intertwined.
  - CP violation established in quark sector, consistent with SM (CKM). ✓
  - open question: CP violation in lepton sector ?
  - open question: Why $\overline{\theta} < 10^{-10}$?
    Why CP violation *only* in FV processes?
- *The* theory of flavor should also be *the* theory of CPV.
- Plan: be humble, try to understand origin of CPV ("only" one parameter).
- The 3HDM with $\mathbf{\Delta(27)}$ symmetry has very interesting CP properties.

Outline

The model: 3HDM with $\boldsymbol{\Delta(27)}$

Spontaneous geometrical CP violation

What is an outer automorphism?

Outer automorphisms in 3HDM with $\boldsymbol{\Delta(54)}$

Summary

The model

3HDM model with $[\boldsymbol{\Delta(27)} \Rightarrow]$ $\boldsymbol{\Delta(54)}$ symmetry.

(This is the original “geometrical T violation” model of Branco, Gerard, and Grimus.) [Branco, Gerard, Grimus, ’83]

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- Three–Higgs potential invariant under $\Delta(54)$, generated by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, C = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

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- "Traditional" way to write the potential: $(i, j = 1, .., 3;\ i \neq j)$

$$V = -m^2 H_i^\dagger H_i + \lambda_1 \left(H_i^\dagger H_i\right)^2 + \lambda_2 \left(H_i^\dagger H_i\right) \left(H_j^\dagger H_j\right) + \lambda_3 \left(H_i^\dagger H_j\right) \left(H_j^\dagger H_i\right)$$
$$+ \mathrm{e}^{\mathrm{i}\,\Omega} \lambda_4 \left[\left(H_1^\dagger H_2\right) \left(H_1^\dagger H_3\right) + \text{cyclic}\right] + \mathrm{h.c.}\ .$$

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- Notation:

$$\langle 0|\, H_i \,|0\rangle \equiv \langle H_i\rangle := \begin{pmatrix} 0 \\ \mathrm{v}_i\,\mathrm{e}^{\mathrm{i}\,\varphi_i} \end{pmatrix} \quad \text{for } i = 1, .., 3$$
$$\langle H\rangle = (\mathrm{v}_1\,\mathrm{e}^{\mathrm{i}\,\varphi_1}, \mathrm{v}_2\,\mathrm{e}^{\mathrm{i}\,\varphi_2}, \mathrm{v}_3\,\mathrm{e}^{\mathrm{i}\,\varphi_3})^{\mathrm{T}}\,.$$

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- Potential gives rise to four classes of VEVs: $v_i = \dfrac{m}{\sqrt{2(a_0 + a_i)}}$, $\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3}$

$$\langle H\rangle_{\mathrm{I}} = v_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \langle H\rangle_{\mathrm{II}} = v_2 \begin{pmatrix} \omega \\ 1 \\ 1 \end{pmatrix}, \langle H\rangle_{\mathrm{III}} = v_3 \begin{pmatrix} \omega^2 \\ 1 \\ 1 \end{pmatrix}, \langle H\rangle_{\mathrm{IV}} = v_4 \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}.$$

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Spontaneous geometrical CP violation

If a CP transformation $H \mapsto U H^*$ is a symmetry of the Lagrangian, then

$$\langle H\rangle \;\neq\; U\,\langle H\rangle^*$$

must hold in order for this CP transformation to be spontaneously violated. [Branco et al. '83]

- For example: $U = \mathbb{1}$ is a CP symmetry if $\Omega = 0, \pi$.
  It is broken by VEVs of type II and III.
  $\curvearrowright$ there appears a physical CPV phase: $\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$.
- All possible forms of U are given by solutions to the "consistency condition" (for various u's)

$$U\,\rho_{\mathbf{3^*}}(g)\,U^\dagger \;=\; \rho_{\mathbf{3}}(u(g)).$$

[Holthausen, Lindner, Schmidt, '13; Feruglio, Hagedorn, Ziegler, '13]

$\Rightarrow$ Actually: CP transformations are special outer automorphism transformations of all present symmetries (in particular $\mathbf{\Delta(54)}$).

What is an outer automorphism?

Example: $\mathbb{Z}_3$ symmetry, generated by $\mathsf{a}^3 = \mathsf{id}$.

$\mathbb{Z}_3$	id	a	a^2
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	ω	ω^2
$\mathbf{1}''$	1	ω^2	ω

$(\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3})$

- All elements of $\mathbb{Z}_3$: $\{\mathsf{id},\, \mathsf{a},\, \mathsf{a}^2\}$.
- Outer automorphism group ("Out") of $\mathbb{Z}_3$: generated by

$$u(\mathsf{a}) : \mathsf{a} \mapsto \mathsf{a}^2. \quad \bigl(\text{think: } \mathsf{u}\,\mathsf{a}\,\mathsf{u}^{-1} \;=\; \mathsf{a}^2\bigr)$$

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Abstract: Out is a reshuffling of symmetry elements. $(\mathrm{Out} \;:=\; {}^{\mathrm{Aut}}/_{\mathrm{Inn}})$
A **"symmetry of the symmetry"**.

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Abstract: Out is a reshuffling of symmetry elements. $(\mathrm{Out} := {}^{\mathrm{Aut}}/_{\mathrm{Inn}})$
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Concrete: Out is a mapping between representations $\boldsymbol{r} \mapsto \boldsymbol{r}'$.
The transformation matrix U is given by the solution to

$$U\rho_{\boldsymbol{r}'}(\mathrm{g})U^{-1} = \rho_{\boldsymbol{r}}(u(\mathrm{g}))\ , \qquad \forall \mathrm{g} \in G\ .$$

(Note: $\boldsymbol{r}' = \boldsymbol{r}^*$ is a special case of this). [Fallbacher, AT, '15]

Outer automorphisms of $\boldsymbol{\Delta(54)}$

Outer automorphisms of a discrete group are symmetries of the character table.

$\Delta(54)$	C_{1a}	C_{3a}	C_{3b}	C_{3c}	C_{3d}	C_{2a}	C_{6a}	C_{6b}	C_{3e}	C_{3f}
$\mathbf{1}_0$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	1	1	1	-1	-1	-1	1	1
$\mathbf{2}_1$	2	2	-1	-1	-1	0	0	0	2	2
$\mathbf{2}_2$	2	-1	2	-1	-1	0	0	0	2	2
$\mathbf{2}_3$	2	-1	-1	2	-1	0	0	0	2	2
$\mathbf{2}_4$	2	-1	-1	-1	2	0	0	0	2	2
$\mathbf{3}_1$	3	0	0	0	0	1	ω^2	ω	3ω	$3\omega^2$
$\overline{\mathbf{3}}_1$	3	0	0	0	0	1	ω	ω^2	$3\omega^2$	3ω
$\mathbf{3}_2$	3	0	0	0	0	-1	$-\omega^2$	$-\omega$	3ω	$3\omega^2$
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- **But:** <u>not</u> all outer automorphisms are CP transformations!

[Chen, Fallbacher, Mahanthappa, Ratz, AT, '14]
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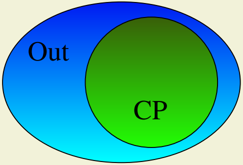

$$\mathrm{Out}:\ \boldsymbol{r}_i \ \mapsto\ \boldsymbol{r}_j$$
$$\mathrm{CP}:\ \boldsymbol{r} \ \mapsto\ \boldsymbol{r}^*$$

Outer automorphisms in $\Delta(54)$ 3HDM

Note: All quartic interactions arise from

$$\left[\left(H_{\mathbf{\overline{3}}}^\dagger \otimes H_{\mathbf{3}}\right) \otimes \left(H_{\mathbf{\overline{3}}}^\dagger \otimes H_{\mathbf{3}}\right)\right]_{\mathbf{1}_0} ,$$

either via $\left[(\mathbf{\overline{3}} \otimes \mathbf{3})_{\mathbf{1}_0} \otimes (\mathbf{\overline{3}} \otimes \mathbf{3})_{\mathbf{1}_0}\right]_{\mathbf{1}_0}$ or via $\left[(\mathbf{\overline{3}} \otimes \mathbf{3})_{\mathbf{2}_i} \otimes (\mathbf{\overline{3}} \otimes \mathbf{3})_{\mathbf{2}_i}\right]_{\mathbf{1}_0}$.

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$\curvearrowright$ A more "natural" way to write the potential:

$$V(H,\vec{a}) \;=\; -\,m^2\,H_i^\dagger H_i + a_0\,I_0(H^\dagger,H) \begin{array}{l} +\,a_1\;\boxed{I_1(H^\dagger,H)}\;+\,a_2\;\boxed{I_2(H^\dagger,H)} \\[2ex] +\,a_3\;\boxed{I_3(H^\dagger,H)}\;+\,a_4\;\boxed{I_4(H^\dagger,H)}\;, \end{array}$$

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Large outer automorphism group:

$$\mathrm{Out}(\Delta(54))=\mathrm{S}_4,\quad \text{maps } H\to UH \text{ (even) or } H\to UH^* \text{ (odd).}$$

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Outer automorphism group applied to the VEVs:

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\langle H\rangle_{\mathrm{I}} = v_1(1,1,1), \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\mathrm{Out}"] \arrow[dr, leftrightarrow] & \langle H\rangle_{\mathrm{II}} = v_2(\omega,1,1), \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
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$$\mathrm{Out}(\Delta(54)) = \mathrm{S}_4, \quad \langle H\rangle \to U\langle H\rangle \text{ (even) or } \langle H\rangle \to U\langle H\rangle^* \text{ (odd).}$$

Four classes of VEVs:

$$\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3},\ v_i = \frac{m}{\sqrt{2(a_0+a_i)}},$$

$$\begin{array}{rcl c rcl}
\langle H\rangle_{\mathrm{I}} &=& v_1(1,1,1), & \longleftrightarrow & \langle H\rangle_{\mathrm{II}} &=& v_2(\omega,1,1), \\
 & & \updownarrow \mathrm{Out} & \begin{smallmatrix}\nwarrow & \nearrow \\ \swarrow & \searrow\end{smallmatrix} & & & \updownarrow \\
\langle H\rangle_{\mathrm{III}} &=& v_3(\omega^2,1,1), & \leftrightarrow & \langle H\rangle_{\mathrm{IV}} &=& v_4(\sqrt{3},0,0).
\end{array}$$

Insights:

- Couplings $\{a_1,a_2,a_3,a_4\}$ form a $\mathbf{4}$–plet under S_4.
  $\Rightarrow$ physically degenerate parameter space $(a_1,a_2,a_3,a_4) \to a_1 \leq a_2 \leq a_3 \leq a_4$.
- VEVs $\Phi := (\langle H\rangle_{\mathrm{I}}, \langle H\rangle_{\mathrm{II}}, \langle H\rangle_{\mathrm{III}}, \langle H\rangle_{\mathrm{IV}})$ form a $\mathbf{4}$–plet under S_4.
  $\Rightarrow$ **VEVs are calculable** from a homogeneous linear equation $M\Phi = 0$.
- This completely **fixes** the directions, and relative (physical) phases of the VEVs.
- VEVs break to ismorphic subgroups $\to$ each VEV encodes the same physics!

[Fallbacher, AT, '15]

Outer automorphisms in $\mathbf{\Delta(54)}$ 3HDM

This "derived" parametrization simplifies the understanding of CP and symmetry enhancement.

CP–odd basis invariant:

$$I_6 \;=\; -9\sqrt{3}(a_1-a_2)(a_1-a_3)(a_1-a_4)(a_2-a_3)(a_2-a_4)(a_3-a_4)\;. \qquad \text{[Nishi]}$$

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  $\curvearrowright$ Require order 2 CP as a symmetry. For example:

  (a) $a_1 < a_2 < a_3 = a_4 \Rightarrow$ global minimum conserves CP

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  (We "add" the corresponding Out to the symmetry group).

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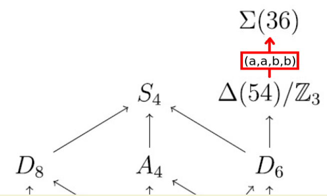

Outer automorphisms in general (beyond C,P)

This 3HDM model is an example for some very general statements on outer automorphisms: [Fallbacher, AT, '15], [AT '16]

- Outs allow to identify physically redundant regions in the parameter space.

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- Outs can give rise to emergent symmetries.

  here: $U\,\langle H\rangle_{\mathrm{I}} \;=\; \langle H\rangle_{\mathrm{I}}$, where $U \,\in\, \mathrm{Out}(G)$.

  $\Rightarrow$ VEV has higher symmetry than potential.

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  $\Rightarrow$ VEV has higher symmetry than potential.
- Ultimately, the direction and relative phases of the VEVs are fixed, because each VEV has to be an eigenvector of some element of the outer automorphism group.
  This is at the heart of (spontaneous) geometrical CP violation.

Summary

- Outer automorphisms are the non–trivial symmetries of a symmetry ($\rightarrow$ think of them as mappings among the irreps).
- CP is **a** special outer automorphism which maps *all* present representations to their complex conjugate representation.
- Outer automorphisms in general:
  - Act as permutation of symmetry invariants,

  $\Rightarrow$ point to physical degeneracies in the parameter space.
  - Can give rise to VEVs with emergent symmetry,

  $\Rightarrow$ allow for a very simple calculation of VEVs.
- In the 3HDM with $\boldsymbol{\Delta(54)}$ $[\boldsymbol{\Delta(27)}]$ symmetry, the large outer automorphism group can be viewed as the reason for spontaneous geometrical CP violation with calculable phases.

Thank You!

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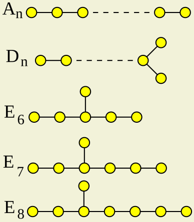
Backup slides

What is an outer automorphism?

☺ There are easy ways to depict this...

Continuous groups:

Outer automorphisms of a Lie algebra are the symmetries of the corresponding Dynkin diagram.



	Lie Group	Out	Action on reps
$A_{n>1}$	$\mathrm{SU}(N)$	$\mathbb{Z}_2$	$\boldsymbol{r} \to \boldsymbol{r}^*$
$D_{n=4}$	$\mathrm{SO}(8)$	S_3	$\boldsymbol{r}_i \to \boldsymbol{r}_j$
$D_{n>4}$	$\mathrm{SO}(2N)$	$\mathbb{Z}_2$	$\boldsymbol{r} \to \boldsymbol{r}^*$
E_6	E_6	$\mathbb{Z}_2$	$\boldsymbol{r} \to \boldsymbol{r}^*$
all others		/	/

What is an outer automorphism?

Discrete groups:

Outer automorphisms of a discrete group are the symmetries of the character table.

$\Delta(54)$	C_{1a}	C_{3a}	C_{3b}	C_{3c}	C_{3d}	C_{2a}	C_{6a}	C_{6b}	C_{3e}	C_{3f}
$\mathbf{1}_0$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	1	1	1	-1	-1	-1	1	1
$\mathbf{2}_1$	2	2	-1	-1	-1	0	0	0	2	2
$\mathbf{2}_2$	2	-1	2	-1	-1	0	0	0	2	2
$\mathbf{2}_3$	2	-1	-1	2	-1	0	0	0	2	2
$\mathbf{2}_4$	2	-1	-1	-1	2	0	0	0	2	2
$\mathbf{3}_1$	3	0	0	0	0	1	ω^2	ω	3ω	$3\omega^2$
$\bar{\mathbf{3}}_1$	3	0	0	0	0	1	ω	ω^2	$3\omega^2$	3ω
$\mathbf{3}_2$	3	0	0	0	0	-1	$-\omega^2$	$-\omega$	3ω	$3\omega^2$
$\bar{\mathbf{3}}_2$	3	0	0	0	0	-1	$-\omega$	$-\omega^2$	$3\omega^2$	3ω

Advantage:

The outer automorphisms of any ("small") discrete group can easily be found with `GAP` [GAP] .

Group	Out	Action on reps
$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\boldsymbol{r} \rightarrow \boldsymbol{r}^*$
$\mathrm{A}_{n\neq 6}$	$\mathbb{Z}_2$	$\boldsymbol{r} \rightarrow \boldsymbol{r}^*$
$\mathrm{S}_{n\neq 6}$	/	/
$\Delta(27)$	$\mathrm{GL}(2,3)$	$\boldsymbol{r}_i \rightarrow \boldsymbol{r}_j$
$\Delta(54)$	S_4	$\boldsymbol{r}_i \rightarrow \boldsymbol{r}_j$
$\ldots$		

$\Delta(27)$	C_{1a} 1 e	C_{3a} 3 A	C_{3b} 3 A^2	C_{3c} 3 B	C_{3d} 3 B^2	C_{3e} 3 ABA	C_{3f} 3 BAB	C_{3g} 3 AB	C_{3h} 3 A^2B^2	C_{3i} 1 AB^2ABA	C_{3j} 1 BA^2BAB
$\mathbf{1}_0$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω	1	1
$\mathbf{1}_2$	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2	1	1
$\mathbf{1}_3$	1	ω^2	ω	1	1	ω	ω^2	ω^2	ω	1	1
$\mathbf{1}_4$	1	ω^2	ω	ω^2	ω	1	1	ω	ω^2	1	1
$\mathbf{1}_5$	1	ω^2	ω	ω	ω^2	ω^2	ω	1	1	1	1
$\mathbf{1}_6$	1	ω	ω^2	1	1	ω^2	ω	ω	ω^2	1	1
$\mathbf{1}_7$	1	ω	ω^2	ω^2	ω	ω	ω^2	1	1	1	1
$\mathbf{1}_8$	1	ω	ω^2	ω	ω^2	1	1	ω^2	ω	1	1
$\mathbf{3}$	3	0	0	0	0	0	0	0	0	$3\omega^2$	3ω
$\overline{\mathbf{3}}$	3	0	0	0	0	0	0	0	0	3ω	$3\omega^2$

Tabelle: Character table of $\Delta(27)$. We define $\omega := \mathrm{e}^{2\pi\,\mathrm{i}/3}$. The conjugacy classes (c.c.) are labeled by the order of their elements and a letter. The second line gives the cardinality of the corresponding c.c. and the third line gives a representative of the c.c. in the presentation specified in the text.

3HDM dictionary

A more "natural" way to write the potential:

$$
\begin{aligned}
\left[\left(H^\dagger_{\bar{\mathbf{3}}} \otimes H_{\mathbf{3}}\right) \otimes \left(H^\dagger_{\bar{\mathbf{3}}} \otimes H_{\mathbf{3}}\right)\right]_{\mathbf{1}_0} \;=\;& a_0 \left[\left(H^\dagger \otimes H\right)_{\mathbf{1}_0} \otimes \left(H^\dagger \otimes H\right)_{\mathbf{1}_0}\right] \\
&+ \frac{a_1}{\sqrt{2}} \left[\left(H^\dagger \otimes H\right)_{\mathbf{2}_1} \otimes \left(H^\dagger \otimes H\right)_{\mathbf{2}_1}\right]_{\mathbf{1}_0} + \frac{a_2}{\sqrt{2}} \left[\left(H^\dagger \otimes H\right)_{\mathbf{2}_3} \otimes \left(H^\dagger \otimes H\right)_{\mathbf{2}_3}\right]_{\mathbf{1}_0} \\
&+ \frac{a_3}{\sqrt{2}} \left[\left(H^\dagger \otimes H\right)_{\mathbf{2}_4} \otimes \left(H^\dagger \otimes H\right)_{\mathbf{2}_4}\right]_{\mathbf{1}_0} + \frac{a_4}{\sqrt{2}} \left[\left(H^\dagger \otimes H\right)_{\mathbf{2}_2} \otimes \left(H^\dagger \otimes H\right)_{\mathbf{2}_2}\right]_{\mathbf{1}_0} .
\end{aligned}
$$

Relations between the two different bases:

$$
\begin{aligned}
3\,\lambda_1 \;&=\; a_0 + a_4 \;, \quad 3\,\lambda_2 \;=\; 2a_0 - a_4 \;, \quad 3\,\lambda_3 \;=\; a_1 + a_2 + a_3 \;, \\
3\,\lambda_4 \;&=\; \left|a_1 + \omega^2\, a_2 + \omega\, a_3\right| \;, \quad \text{and} \quad \Omega \;=\; \arg\left(a_1 + \omega^2\, a_2 + \omega\, a_3\right) \;.
\end{aligned}
$$

Bounded–below criterions:

$$
0 \;<\; \lambda_1 \qquad \text{and} \qquad 0 \;<\; \lambda_1 + \lambda_{23} + 2\,\lambda_4\,\cos\left[2\pi/3 + (\Omega \mod 2\pi/3)\right] \;,
$$

vs.

$$
0 \;<\; a_0 + a_\ell \;, \qquad \text{for}\;\; \ell = 1,..,4 \;.
$$

Invariants spelled out:

$$I_0(H^\dagger, H) \;=\; \frac{1}{3}\left(H_1^\dagger H_1 + H_2^\dagger H_2 + H_3^\dagger H_3\right)^2 \;,$$

$$\begin{aligned} I_1(H^\dagger, H) \;=\; \frac{\sqrt{2}}{3}\Big[&\Big(H_1^\dagger H_2 H_1^\dagger H_3 + H_2^\dagger H_1 H_2^\dagger H_3 + H_3^\dagger H_1 H_3^\dagger H_2 + \mathrm{h.c.}\Big) + \\ & H_1^\dagger H_2 H_2^\dagger H_1 + H_1^\dagger H_3 H_3^\dagger H_1 + H_2^\dagger H_3 H_3^\dagger H_2\Big] \;, \end{aligned}$$

$$\begin{aligned} I_2(H^\dagger, H) \;=\; \frac{\sqrt{2}}{3}\Big[&H_1^\dagger H_1 H_1^\dagger H_1 + H_2^\dagger H_2 H_2^\dagger H_2 + H_3^\dagger H_3 H_3^\dagger H_3 \\ & -H_1^\dagger H_1 H_2^\dagger H_2 + -H_1^\dagger H_1 H_3^\dagger H_3 + -H_2^\dagger H_2 H_3^\dagger H_3\Big] \;, \end{aligned}$$

$$\begin{aligned} I_3(H^\dagger, H) \;=\; \frac{\sqrt{2}}{3}\Big[&\Big(\omega^2 H_1^\dagger H_2 H_1^\dagger H_3 + \omega^2 H_2^\dagger H_1 H_2^\dagger H_3 + \omega^2 H_3^\dagger H_1 H_3^\dagger H_2 + \mathrm{h.c.}\Big) + \\ & H_1^\dagger H_2 H_2^\dagger H_1 + H_1^\dagger H_3 H_3^\dagger H_1 + H_2^\dagger H_3 H_3^\dagger H_2\Big] \;, \end{aligned}$$

$$\begin{aligned} I_4(H^\dagger, H) \;=\; \frac{\sqrt{2}}{3}\Big[&\Big(\omega H_1^\dagger H_2 H_1^\dagger H_3 + \omega H_2^\dagger H_1 H_2^\dagger H_3 + \omega H_3^\dagger H_1 H_3^\dagger H_2 + \mathrm{h.c.}\Big) + \\ & H_1^\dagger H_2 H_2^\dagger H_1 + H_1^\dagger H_3 H_3^\dagger H_1 + H_2^\dagger H_3 H_3^\dagger H_2\Big] \;. \end{aligned}$$

If there is an outer automorphism transformation u acting consistently with the symmetries and representations of a model then it is possible to obtain new VEVs from a known one $\langle\Phi(\lambda)\rangle$ simply by taking

$$\langle\Phi(\lambda)\rangle_{\text{new}} = \begin{cases} U\,\langle\Phi\,(\lambda\to\lambda')\rangle\;, & \text{if} \quad u\;:\;\boldsymbol{r}_\Phi\;\mapsto\;U\,\boldsymbol{r}_\Phi\;,\ \text{or} \\ U\,\langle\Phi\,(\lambda\to\lambda')\rangle^*\;, & \text{if} \quad u\;:\;\boldsymbol{r}_\Phi\;\mapsto\;U\,\boldsymbol{r}_\Phi^*\;. \end{cases} \tag{1}$$

This implies that stationary points of potentials always appear in complete multiplets of the available group of outer automorphisms.