Some lessons on bounded-from-below conditions in multi-Higgs models

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based on: I. P. Ivanov, F. Faro, PRD100 (2019) 035038
I. P. Ivanov, F. Vazão, JHEP 11 (2020) 104 [covid lockdown paper]
N. Buskin, I. P. Ivanov, J.Phys.A 54 (2021) 325401

and numerous discussions with J.P. Silva.

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N-Higgs-doublet models

Higgses can come in generations $\rightarrow$ NHDMs [T.D.Lee 1973, Weinberg 1976, ...]

SM

u up	c charm	t top	γ photon
d down	s strange	b bottom	Z Z boson
ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson
e electron	μ muon	τ tau	g gluon

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Multi-Higgs models

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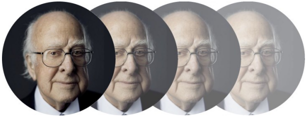

3HDM vs 2HDM

Model building: more options for scalar and fermion sectors $\Rightarrow$ richer pheno

- Many more symmetry group choices [classic papers], exact or approximate; that's the single most powerful novelty of the 3HDMs.
- Some pheno consequences driven by structural features rather than numerical values of coefs, such as
  - geometrical *CP* violation [Branco, Gerard, Grimus, 1984];
  - *CP* symmetry of order 4 [Ivanov, Silva, 2015];
  - 3HDM with softly broken $\Sigma(36)$ [Varzielas, Ivanov, Levy, 2021];
- combining features of 2HDM: e.g. NFC + CPV [Weinberg, 1976; Branco, 1979], scalar DM + CPV [Grzadkowski et al, 2009].
- astroparticle consequences: richer dark sectors [Cordero et al, 2017]; new options for baryon asymmetry [Davoudiasl, Lewis, Sullivan, 2019]; many minima $\rightarrow$ multi-step phase transitions.

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NHDM in a nutshell

N Higgs doublets ϕ_a, $a = 1, \ldots, N$, with equal quantum numbers.

- The general NHDM potential

$$V = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{ab,cd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,,$$

  with $N^2(N^2+3)/2$ free parameters (14 for the 2HDM, 54 for the 3HDM).

- The quark Yukawa sector $\bar{Q}_{Li}\Gamma_{ij}^{(a)}\phi_a d_{Rj} + \bar{Q}_{Li}\Delta_{ij}^{(a)}\tilde{\phi}_a u_{Rj} + h.c.$ leads to m_q, V_{CKM}, FCNCs, production and decay patterns, etc.

BFB conditions as a particular aspect of NHDMs

When working out pheno of a multi-Higgs model, make sure you do your maths

- correctly,
- efficiently.

Here, I focus on a particular issue concerning multi-Higgs potentials: the bounded-from-below conditions (BFB conditions).

$$\text{Given} \quad V = V_2 + V_4 = Y_{ab}(\phi_a^\dagger \phi_b) + Z_{ab,cd}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,,$$

what are the necessary and sufficient conditions the coefs need to satisfy for the potential to be bounded from below?

That's essentially the condition $V_4 \geq 0$ everywhere in the ϕ_a space.

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BFB conditions in NHDMs: what's known

General results:

- BFB conditions in the general 2HDM [Ivanov, 2007] based on the bilinear formalism [Maniatis et al, 2004–2007; Ivanov, 2006–2007; Nishi, 2006–2008].
- Although the bilinear formalism is easily extended to NHDM, applying it to the BFB problem is very challenging beyond 2HDM.
- A general algebraic approach based on spectral theory of order-4 tensors was proposed [Ivanov, Köpke, Mühlleitner, 2018], not yet practical.
- A generic lesson: do not expect the BFB conditions to always be of the form $\lambda_i \geq \dots$. Solving equations numerically seems unavoidable at some stage.
- The hope is to reduce the problem to a single algebraic eqn on a single variable with analytically known coefs (e.g. a characteristic equation).
- Perhaps [Maniatis, Nachtmann, 1408.6833] already presents a procedure to follow.

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Specific results in 3HDM:

- If $V(\phi_a)$ depends only on $x_a = |\phi_a|^2$, then the $V_4(x_a)$ is a quadratic form of independent $x_a \geq 0 \to$ copositivity conditions are applicable [Kannike, 2012].
- For 3HDMs invariant under large symmetry groups (e.g. $\Delta(27)$, $\Sigma(36)$), very few coefs remain, and the BFB conditions can be found analytically via a geometric method developed in [Degee, Ivanov, Keus 2012; Ivanov, Nishi 2015].
- For smaller symmetry groups such as S_4, A_4, $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Weinberg's model) etc., the situation remained confusing: several conflicting versions were published and used in pheno analyses.
- We tried to clarify the situation, at least for
  - $U(1) \times U(1)$ 3HDM, [Faro, Ivanov, 1907.01963];
  - S_4 and A_4 3HDM, [Ivanov, Vazão, 2006.00036], [Buskin, Ivanov, 2104.11428].

Below, I report on our main findings.

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The importance of charge-breaking directions

Rephasing-invariant 3HDM

The $U(1)\times U(1)$-symmetric potential is $V = V_2 + V_N + V_{CB}$, where
$V_2 = m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{33}^2(\phi_3^\dagger\phi_3)$ and

$$\begin{aligned} V_N &= \frac{\lambda_{11}}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_{22}}{2}(\phi_2^\dagger\phi_2)^2 + \frac{\lambda_{33}}{2}(\phi_3^\dagger\phi_3)^2 \\ &\quad +\lambda_{12}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_{13}(\phi_1^\dagger\phi_1)(\phi_3^\dagger\phi_3) + \lambda_{23}(\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3)\,, \\ V_{CB} &= \lambda'_{12}z_{12} + \lambda'_{13}z_{13} + \lambda'_{23}z_{23}\,. \end{aligned}$$

Here, $z_{ab} = |\phi_a|^2|\phi_b|^2 - |\phi_a^\dagger\phi_b|^2 \geq 0$.
"Neutral" directions: $\phi_1 \propto \phi_2 \propto \phi_3 \to z_{ab} = 0$.

We call all other directions in the Higgs space the "charge-breaking" directions.

If we add $\bar\lambda_{12}(\phi_1^\dagger\phi_2)^2 + \bar\lambda_{13}(\phi_1^\dagger\phi_3)^2 + \bar\lambda_{23}(\phi_2^\dagger\phi_3)^2 + h.c.$, we get Weinberg's or Branco's models ($\mathbb{Z}_2\times\mathbb{Z}_2$ with explicit CPV or CPC).

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BFB conditions within the neutral space

Suppose we have a $U(1) \times U(1)$ 3HDM with a neutral vacuum, with all physical Higgses (including $H^\pm$) having $m^2_{H_i} > 0$.

It is tempting to derive the BFB conditions for this model working only within the neutral space ($z_{ab} = 0$).

Define $r_a = |\phi_a|^2 \geq 0$, write $V_N = A_{ab} r_a r_b / 2$, and apply the copositivity conditions.

But is it really safe to bypass the BFB check along the charge-breaking directions?

In [Faro, Ivanov, 1907.01963], we answered this question in the negative.

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BFB conditions within the charge-breaking space

- First, fix $r_a = |\phi_a|^2$ and extremize

$$V_{CB} = \lambda'_{12} z_{12} + \lambda'_{13} z_{13} + \lambda'_{23} z_{23}$$

with respect to relative "angles" among the doublets (reminder: $z_{ab} = |\phi_a|^2|\phi_b|^2 - |\phi_a^\dagger \phi_b|^2$).

- Extrema at non-trivial angles exist if $r_1/|\lambda'_{23}|$, $r_2/|\lambda'_{31}|$, and $r_3/|\lambda'_{12}|$ satisfy the triangle inequalities, leading to

$$V_{CB}^{\mathrm{non-triv.}} = \frac{\lambda'_{12}\lambda'_{13}\lambda'_{23}}{4}\left(\frac{r_1}{\lambda'_{23}} + \frac{r_2}{\lambda'_{13}} + \frac{r_3}{\lambda'_{12}}\right)^2 .$$

- Thus, if $\lambda'_{12}\lambda'_{13}\lambda'_{23} < 0$, there exists a ray in the charge-breaking space along which the potential, asymptotically, is than anywhere in the neutral space. One must check whether $V_4 \geq 0$ along this direction!
- This can be done, again, via the copositivity methods (accounting for the linear relations among r_a), see [Faro, Ivanov, 1907.01963].
- Extended to the $U(1) \times \mathbb{Z}_2$ 3HDM in [Faro, MSc thesis, 2019].

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The importance of charge-breaking directions

The lesson

even if you have a model with a neutral minimum and $m^2 > 0$ for all physical Higgses, you still must check the BFB conditions along all Higgs space directions, including the charge-breaking ones.

The remark applies to other multi-Higgs models such as 331 model (3 triplets) [Costantini, Ghezzi, Pruna, 2020] or with $SU(n)$ scalars [Fonseca, 2021].

The surprising importance of soft symmetry-breaking terms

The S_4 symmetric 3HDM

First considered in [Pakwasa, Sugawara, 1979]. Following [Degee, Ivanov, Keus, 1211.4989], we write it as

$$
\begin{aligned}
V &= -M_0 R + \Lambda_0 R^2 + \frac{\Lambda_3}{3}\left(r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_3 r_1\right) \\
&+ \Lambda_1\left(\mathrm{Re}_{12}^2 + \mathrm{Re}_{23}^2 + \mathrm{Re}_{31}^2\right) + \Lambda_2\left(\mathrm{Im}_{12}^2 + \mathrm{Im}_{23}^2 + \mathrm{Im}_{31}^2\right).
\end{aligned}
$$

where $r_a \equiv |\phi_a|^2$, $R \equiv (r_1 + r_2 + r_3)/3$, and $\phi_a^\dagger \phi_b \equiv \mathrm{Re}_{ab} + i\,\mathrm{Im}_{ab}$.

Take out R^2 in all the quartic terms:

$$V = -M_0 R + R^2\left(\Lambda_0 + \Lambda_1 x + \Lambda_2 y + \Lambda_3 z\right), \quad \text{where}$$

$$x = \frac{\sum \mathrm{Re}_{ab}^2}{R^2}, \quad y = \frac{\sum \mathrm{Im}_{ab}^2}{R^2}, \quad z = \frac{r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_3 r_1}{3R^2},$$

The potential is a linear function of x, y, z, which are subject to constraints.

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The S_4 symmetric 3HDM

The BFB condition for the S_4 3HDM is

$$v_4 \equiv \Lambda_0 + \Lambda_1 x + \Lambda_2 y + \Lambda_3 z > 0$$

everywhere in the definition domain of (x, y, z).

What is the shape of this domain?

The orbit space of the S_4 3HDM

$$x = \frac{\sum \mathrm{Re}_{ab}^2}{R^2}\,, \quad y = \frac{\sum \mathrm{Im}_{ab}^2}{R^2}\,, \quad z = \frac{r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_3 r_1}{3R^2}\,,$$

- From the definitions: $0 \le x, y, z \le 1$.
- Neutral directions: $x + y + z = 1$.
- Charge-breaking directions: $1/4 \le x + y + z < 1$.

$$x = \frac{\sum \mathrm{Re}^2_{ab}}{R^2}\,, \quad y = \frac{\sum \mathrm{Im}^2_{ab}}{R^2}\,, \quad z = \frac{r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_3 r_1}{3R^2}\,,$$

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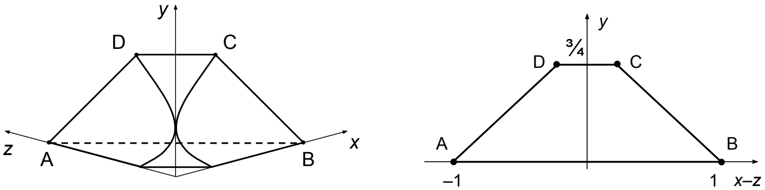
The overall shape was found back in [Degee, Ivanov, Keus, 1211.4989]:



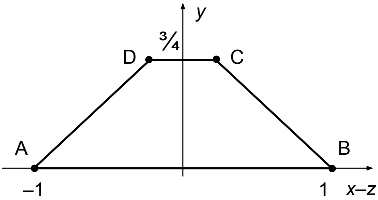
The exact shape of the arcs was found in [Ivanov, Vazão, 2006.00036].

BFB conditions in the neutral orbit space

Then, the BFB conditions for the S_4 3HDM within the neutral space easily follow: we require that the linear function be positive

$$v_4 = \Lambda_0 + \frac{\Lambda_1 + \Lambda_3}{2} + \left(\Lambda_2 - \frac{\Lambda_1 + \Lambda_3}{2}\right) y + \frac{\Lambda_1 - \Lambda_3}{2}(x - z) > 0$$

everywhere in a convex polygon.



Enough to require $v_4 > 0$ at the four vertices, [Degee, Ivanov, Keus, 1211.4989]:

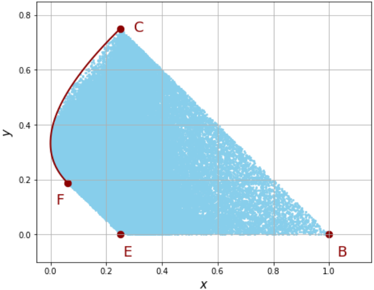
$$\Lambda_0 + \Lambda_3 \geq 0\,, \quad \Lambda_0 + \Lambda_1 \geq 0\,, \quad \Lambda_0 + \frac{\Lambda_1 + 3\Lambda_2}{4} \geq 0\,, \quad \Lambda_0 + \frac{\Lambda_3 + 3\Lambda_2}{4} \geq 0\,.$$

BFB conditions in the charge-breaking orbit space

The exact shape of the charge-breaking part was established only in [Ivanov, Vazão, 2006.00036].

For example, segment FC on the $z = 0$ face:

$$x = (1 - \sqrt{3y})^2\,, \quad \frac{3}{16} \le y \le \frac{3}{4}.$$



It leads to an extra pair of BFB condition coming from the charge-breaking space:

$$\Lambda_0 + \frac{\Lambda_1\Lambda_2}{\Lambda_2 + 3\Lambda_1} \ge 0\,,$$

applicable only when $\Lambda_1 > |\Lambda_2|$, plus a similar condition with $\Lambda_1 \to \Lambda_3$.

Two ways to present the BFB conditions

To summarize: the set of necessary and sufficient BFB conditions for S_4 3HDM is

$$\Lambda_0 + \Lambda_1 \geq 0\,, \quad \Lambda_0 + \frac{\Lambda_1 + 3\Lambda_2}{4} \geq 0\,, \quad \Lambda_0 + \frac{\Lambda_1\Lambda_2}{\Lambda_2 + 3\Lambda_1} \geq 0 \ \ (\text{if } \Lambda_1 > |\Lambda_2|),$$

plus the same with $\Lambda_1 \to \Lambda_3$.

However, if we know that the minimum is neutral and $m^2_{H^\pm_{1,2}} > 0$,
we can safely skip the last condition due to convexity of the orbit space.

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Two ways to present the BFB conditions

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Now add quadratic terms $m^2_{ab}(\phi^\dagger_a\phi_b)$, which break S_4 softly.

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A surprising conclusion

Soft breaking terms may invalidate your BFB conditions if they (implicitly) rely on having a neutral minimum.

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Additional remarks

The BFB conditions for the A_4 3HDM are more involved.

- The answer was found only within the neutral orbit space (the final piece in [Buskin, Ivanov, 2104.11428]).
- A pathological situation was recently found in [Carrolo, Romão, Silva, 2207.02928]: a neutral minimum satisfying neutral BFB conditions, but the potential is unbounded from below along a charge-breaking direction.
- The full set of BFB conditions is still needed for A_4 3HDM.
- The full set of BFB conditions for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3HDM are also not known.

Do you really need to spend time on the exact BFB conditions?

- Well, it depends on what you eventually look for. Finding sufficient conditions can be much easier, and this may be enough for a decent pheno study, see e.g. [Boto, Romão, Silva, 2208.01068].

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Summary

- There exist subtle mathematical pitfalls in multi-Higgs model building. Overlooking them may mislead you. Numerical methods are not always a remedy.
- Bounded-from-below conditions for several versions of the 3HDM present surprises which may defy intuition.