Conditions for the custodial symmetry in multi-Higgs-doublet models

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The custodial symmetry (CS) in the SM

- The custodial symmetry[1] is an approximate symmetry, guards the ρ parameter from large radiative corrections:

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2(\vartheta_W)} \tag{1}$$

- On tree-level $\rho = 1$
- If CS was exact $\Rightarrow$ $\rho = 1$ at all orders of perturbation theory. (But broken by kinetic Higgs terms and Yukawas)
- $\rho = 1.01019 \pm 0.00009$ when $\cos^2(\vartheta_W)$ is interpreted in $\overline{\text{MS}}$ at energy scale m_Z.[2]

[1] P. Sikivie, L. Susskind, M. B. Voloshin and V. I. Zakharov, Nucl. Phys. B **173** (1980) 189.
[2] R. L. Workman *et al.* [Particle Data Group], PTEP **2022** (2022), 083C01

CS is exact in the SM Higgs potential:

- SM Higgs Lagrangian:

$$\mathcal{L}_{\mathcal{H}} = (D_\alpha \Phi)^\dagger (D^\alpha \Phi) - V_{\mathsf{SM}}(\Phi) \tag{2}$$

- where $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ v + \phi_3 + i\phi_4 \end{pmatrix}$, covar. derivative $D^\alpha = \partial^\alpha + \frac{ig}{2}\sigma_j W_j^\alpha + \frac{ig'}{2} B^\mu$, and potential

$$V_{\mathsf{SM}}(\Phi) = \lambda(\Phi^\dagger\Phi)^2 + \mu^2\Phi^\dagger\Phi. \tag{3}$$

- Write Φ as a real quadruplet $\Phi_r = \begin{pmatrix} \mathrm{Re}(\Phi) \\ \mathrm{Im}(\Phi) \end{pmatrix}$, then $V_{\mathsf{SM}}(\Phi) = V_{\mathsf{SM}}(\Phi_r)$ is invariant under

$$\Phi_r \to O\Phi_r, \quad O \in O(4). \tag{4}$$

- In the limit $g' \to 0$ ($g' = U(1)_Y$ hypercharge coupling) the kinetic terms, $(D_\alpha \Phi)^\dagger (D^\alpha \Phi)$, are invariant under $SO(4) \subset O(4)$.
- This is the custodial $SO(4)$ symmetry ("$SO(4)_C$"), spontaneously broken down to "custodial" $SO(3)$ in the SM.

The custodial symmetry in the NHDM

- In NHDM ($N \geq 2$), there may also be terms in the potential violating CS, even before SSB.
- $\Rightarrow$ The NHDM potential is not always CS.

- In the 2HDM necessary and sufficient conditions for CS are given in H. E. Haber and D. O'Neil, Phys. Rev. D **83** (2011) 055017 [arXiv:1011.6188] and B. Grzadkowski, M. Maniatis and J. Wudka, JHEP **1111** (2011) 030 [arXiv:1011.5228].
- Necessary and sufficient conditions for 3HDM and necessary conditions for 4HDM and 5HDM are given in C. C. Nishi, Phys. Rev. D **83** (2011) 095005 [arXiv:1103.0252].
- I apply a generalization of B. Grzadkowski, M. Maniatis and J. Wudka's formalism.[3]

[3] M. Maniatis and O. Nachtmann, Phys. Rev. D **92** (2015) no.7, 075017 [arXiv:1504.01736 [hep-ph]].

- The NHDM-potential may be built up by the following hermitian building blocks ("bilinears"):

$$\begin{aligned} \widehat{B}_{mn} &\equiv \frac{1}{2}(\Phi_m^\dagger \Phi_n + \Phi_n^\dagger \Phi_m) = \mathrm{Re}(\Phi_m^\dagger \Phi_n), \\ \widehat{A}_m &\equiv \widehat{B}_{mm} \\ \widehat{C}_{mn} &\equiv \frac{-i}{2}(\Phi_m^\dagger \Phi_n - \Phi_n^\dagger \Phi_m) = \mathrm{Im}(\Phi_m^\dagger \Phi_n). \end{aligned}$$

- Here the $\widehat{B}$'s are $O(4)$-symmetric (i.e. CS), while the $\widehat{C}$'s are not:

Symmetry group of bilinears $\widehat{B}$

$$\widehat{B}_{mn} = \mathrm{Re}(\Phi_m^\dagger \Phi_n) = \left(\mathrm{Re}(\Phi_m)^T, \mathrm{Im}(\Phi_m)^T\right) \begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix}, \quad (5)$$

which is invariant under $O(4)$ transformations O

$$\begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix} \to O \begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix}, \quad O^T O = I_{4\times 4}. \quad (6)$$

Symmetry group of bilinears $\widehat{C}$

Write

$$\widehat{C}_{mn} = \mathrm{Im}(\Phi_m^\dagger \Phi_n) = \left(\mathrm{Re}(\Phi_m)^T, \mathrm{Im}(\Phi_m)^T\right) J \begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix} \quad (7)$$

$$= \mathrm{Re}(\Phi_m)^T \mathrm{Im}(\Phi_n) - \mathrm{Im}(\Phi_m)^T \mathrm{Re}(\Phi_n),$$

where

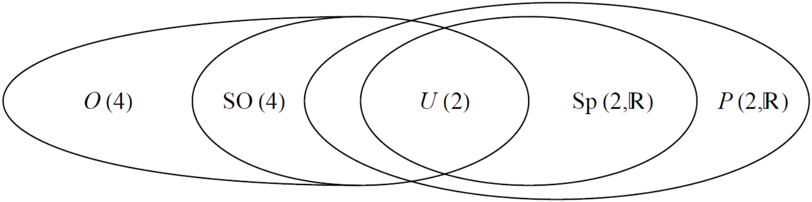
$$J = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2} \end{pmatrix}. \quad (8)$$

$\widehat{C}_{mn}$ is then invariant under the real symplectic group $Sp(2, \mathbb{R})$

$$\begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix} \to S \begin{pmatrix} \mathrm{Re}(\Phi_n) \\ \mathrm{Im}(\Phi_n) \end{pmatrix}, \quad (9)$$

defined by

$$S^T J S = J. \quad (10)$$



- $SO(4)$: Custodial symmetry
- $O(4)$: symmetry group of bilinears $\widehat{B}$
- $Sp(2,\mathbb{R})$: symmetry group of bilinears $\widehat{C}$
- $U(2) \cong SO(4) \cap Sp(2,\mathbb{R}) \cong SU(2)_L \times U(1)_Y$: global symmetry of the SM
- $P(2,\mathbb{R})$: symmetry group of quartic terms $\widehat{C}_{mn}\widehat{C}_{m'n'}$.

- Most general NHDM-potential can then be written (summation over repeated indices)

$$\begin{aligned} V &= \mu_{mn}\widehat{B}_{mn} + \mu^{(2)}_{mn}\widehat{C}_{mn} + \lambda^{(1)}_{mn,m'n'}\widehat{B}_{mn}\widehat{B}_{mn} \\ &+ \lambda^{(2)}_{mn,m'n'}\widehat{B}_{mn}\widehat{C}_{mn} + \lambda^{(3)}_{mn,m'n'}\widehat{C}_{mn}\widehat{C}_{mn}, \end{aligned} \tag{11}$$

- while the most general CS potential is

$$V_{CS} = \mu_{mn}\widehat{B}_{mn} + \lambda^{(1)}_{mn,m'n'}\widehat{B}_{mn}\widehat{B}_{mn}. \tag{12}$$

- In V_{CS} the CS is manifest.
- For a specific potential, the CS may be hidden, that is, not manifest.

- A CS potential can always be transformed into a manifestly CS potential through a $SU(N)$ basis shift.[4]
- We can now apply a "bilinear formalism" to derive necessary and sufficient conditions for having a CS potential:

[4] C. C. Nishi, Phys. Rev. D **83** (2011) 095005 [arXiv:1103.0252 [hep-ph]].

Bilinear formalism

- The general NHDM potential may be written[5]

$$V = \xi_0 K_0 + \xi_a K_a + \eta_0 K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b, \quad (13)$$

- where the N^2 linearly independent bilinears can be written

$$K_\alpha = \text{Tr}(\tilde{K}\lambda_\alpha). \quad (14)$$

  where the Hermitian $N \times N$ matrix $\tilde{K}$ is given by $\tilde{K}_{ij} = \Phi_j^\dagger \Phi_i$ and λ_α are generalized Gell-Mann matrices.
- The ξ's, η's and E_{ab} are parameters.
- We define the matrices λ_α such that the $SO(4)_C$-violating bilinears $\widehat{C}$ are ordered first:

$$K_a = 2\widehat{C}_{m(a),n(a)}, \quad \text{for } 1 \leq a \leq \frac{N(N-1)}{2} \equiv k. \quad (15)$$

[5] M. Maniatis and O. Nachtmann, Phys. Rev. D **92** (2015) no.7, 075017 [arXiv:1504.01736 [hep-ph]].

- In the 3HDM we then get elements K_a, $a \in \{1, 2, \ldots, N^2 - 1\}$ given by

$$\vec{K} = 2\left(\widehat{C}_{12}, \widehat{C}_{13}, \widehat{C}_{23}, \widehat{B}_{12}, \widehat{B}_{13}, \widehat{B}_{23}, \frac{\widehat{A}_1 - \widehat{A}_2}{2}, \frac{\widehat{A}_1 + \widehat{A}_2 - 2\widehat{A}_3}{2\sqrt{3}}\right)^T, \tag{16}$$

- While, generally,

$$K_0 = \mathrm{Tr}(\tilde{K}\lambda_0) = \sqrt{\frac{2}{N}}(\widehat{A}_1 + \ldots + \widehat{A}_N)$$

- The general NHDM potential:

$$V = \xi_0 K_0 + \xi_a K_a + \eta_0 K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b, \quad (13)$$

- Under a Higgs basis shift $\Phi_i \to \Phi'_i = U_{ij}\Phi_j$, V transforms as

$$\begin{aligned} \xi_0 &\to \xi_0, \quad \eta_0 \to \eta_0, \\ \vec{\xi} &\to R(U)\vec{\xi}, \quad \vec{\eta} \to R(U)\vec{\eta}, \\ E &\to E' = R(U) E R^T(U), \end{aligned} \quad (17)$$

- where $R(U) \in \mathrm{Ad}_{SU(N)} \subset SO(N^2-1)$ is given by

$$U^\dagger \lambda_a U = R_{ab}(U)\lambda_b. \quad (18)$$

- The bilinears transform under the adjoint representation of $SU(N)$.
- 2HDM: $\text{Ad}_{SU(2)} = SO(3)$
- NHDM, N>2: $\text{Ad}_{SU(N)} \subsetneq SO(N^2 - 1)$
- ⇒ "Harder" to know when you can transform a potential to a manifestly CS potential in NHDM, N>2, since not all orthogonal matrices are at your disposal.

Main result

$V = \xi_0 K_0 + \xi_a K_a + \eta_0 K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b$. V is $SO(4)_C$-symmetric $\Leftrightarrow$

i) The nullity l of E is $\geq k = N(N-1)/2$.

ii) $\exists$ a real $(N^2-1) \times (N^2-1)$ matrix R whose $k = N(N-1)/2$ first rows are an orthonormal set of nullvectors of E, such that

$$f^{abc} = R_{ai} R_{bj} R_{ck} f^{ijk}, \tag{19}$$

is satisfied for all a, b and c. f^{ijk} here are the structure constants associated with the alternatively ordered, generalized Gell-Mann matrices $\{\lambda_j\}_{j=1}^{N^2-1}$.

iii) R of condition ii) also satisfies

$$R_{ij}\xi_j = 0 \quad \text{and} \quad R_{ij}\eta_j = 0 \quad \text{for all} \quad 1 \leq i \leq \frac{N(N-1)}{2} \equiv k.$$

- Conditions essentially as in the 2HDM, except from one new:
- Existence of a rotation matrix R which rotates E to a manifestly CS form, and where
- $R \in \mathrm{Ad}_{SU(N)} \subsetneq SO(N^2-1) \Leftrightarrow$

$$f^{abc} = R_{ai} R_{bj} R_{ck} f^{ijk}. \tag{19}$$

The case $N = 2$

- $f^{abc} = R_{ai} R_{bj} R_{ck} f^{ijk} \overset{N=2}{\Leftrightarrow}$

$$\begin{aligned} \epsilon^{abc} &= R_{ai} R_{bj} R_{ck} \epsilon^{ijk} \\ &= \det(R) \epsilon^{abc} \end{aligned}$$

- Which holds for any $R \in SO(3)$ ($= \mathrm{Ad}_{SU(2)}$).
- This new condition evaporates in the case N=2.

$N > 2$

- Main problem for determining CS:
- Proving or disproving the existence of a matrix R with the property

$$f^{abc} = R_{ai} R_{bj} R_{ck} f^{ijk} \tag{19}$$

(alternatively, the equivalent $R_{ek} f^{ijk} = R_{ai} R_{cj} f^{ace}$.)

- N=3: Solving 56 ($\sim N^6$) cubic equations in 40 ($\sim N^4$) variables.
- An numerical 3HDM-example, where CS is shown in a certain potential by solving eqs. (19) with Mathematica:

An example

Consider the 3HDM potential given by

$$V = \xi_0 K_0 + \xi_a K_a + \eta_0 K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b,$$

with

$$V = \xi_0 K_0 + \xi_a K_a + \eta_0 K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b,$$

$$E = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 1 & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & 2\sqrt{2} & -1 & 2\sqrt{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 1 & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} \\ 1 & 2\sqrt{2} & 1 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 1 & 2\sqrt{2} & 1 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 & 0 \\ \frac{\sqrt{\frac{3}{2}}}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & 0 & 0 & 0 & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix},$$

$$\vec{\xi} = \left(\frac{1}{6},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{6},\ -\frac{1}{6},\ -\frac{1}{3\sqrt{2}},\ -\frac{1}{6},\ \frac{1}{2\sqrt{6}},\ -\frac{1}{6\sqrt{2}} \right)^T,$$

$$\vec{\eta} = \left(\frac{5\sqrt{2}}{3},\ -\frac{2}{3},\ \frac{5\sqrt{2}}{3},\ -\frac{2\sqrt{2}}{3},\ 2,\ -\frac{2\sqrt{2}}{3},\ \frac{1}{\sqrt{3}},\ -\frac{1}{3} \right)^T,$$

with arbitrary η_0 and ξ_0.

- Nullity$(E) = 3 \geq k = N(N-1)/2 = 3 \Rightarrow$ Condition i) satisfied.
- Mathematica then gives the following orthonormal nullvectors (i.e. eigenvalue 0) of E:

$$\begin{aligned} \tilde{n}_1 &= \left(0, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T, \\ \tilde{n}_2 &= \left(0, 0, 0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0\right)^T, \\ \tilde{n}_3 &= \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0\right)^T. \end{aligned} \tag{20}$$

- We now apply Mathematica's Solve-command, and solve $f^{abc} = R_{ai}R_{bj}R_{ck}f^{ijk}$ (19) with $\tilde{n}_1$, $\tilde{n}_2$ and $\tilde{n}_3$ as the first 3 rows of R.
- We then get a solution

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

- which corresponds to the manifestly $SO(4)_C$-symmetric matrix E':

$$E' = RER^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 4 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad (21)$$

- cf. the terms $K_a E_{ab} K_b$ of V, with $K_{1,2,3} \propto \widehat{C}$.

- $\Rightarrow$ Condition ii) of the main result is satisfied.
- We can then check that $(R\vec{\xi})_i = 0$ and $(R\vec{\eta})_i = 0$ for $i = 1, 2, 3$
- $\Rightarrow$ Condition iii) is satisfied.
- $\Rightarrow V$ is $SO(4)_C$-symmetric.

- Finally, we can then find a Higgs basis transformation $U \in SU(3)$ which corresponds to this R through the relation

$$U^\dagger \lambda_a U = R_{ab}(U)\lambda_b. \qquad (18)$$

- One such matrix will be

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & i & 0 \end{pmatrix}. \qquad (22)$$

- (αU and $\alpha^2 U$ where $\alpha = e^{\frac{2\pi i}{3}}$ will also correspond to the same R.)