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LIE ALGEBRA REPRESENTATIONS AND SYMMETRIES OF NHDM POTENTIALS

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Overview

- Based on (both papers co-authored with Robin Plantey):
- *Computable conditions for order-2 CP symmetry in NHDM potentials*, JHEP 05 (2024) 260, arXiv:2404.02004
- *Representation-theoretical characterization of canonical custodial symmetry in NHDM potentials*, Nuclear Physics B, Volume 1006, 2024, 116650, arXiv:2407.05085
- We will show how it is possible to decide whether or not a NHDM potential has an order-2 CP symmetry (CP2) or a canonical custodial symmetry (CS).
- Done by detecting the defining representation of the Lie algebra $\mathfrak{so}(N)$ for CP2, and certain bases of the defining representation of $\mathfrak{so}(N)$ for CS.

Bilinear formalism

- The general NHDM potential may be written[1]

$$V = M_0 K_0 + M_a K_a + \Lambda_0 K_0^2 + L_a K_0 K_a + \Lambda_{ab} K_a K_b, \tag{1}$$

- where the N^2 linearly independent bilinears can be written

$$K_0 = \Phi_i^\dagger \Phi_i\,, \qquad K_a = \Phi_i^\dagger (\lambda_a)_{ij} \Phi_j. \tag{2}$$

  where λ_α are generalized Gell-Mann matrices.
- The M's, L's and Λ's are parameters.
- We define the GM matrices λ_α such that the $k \equiv N(N-1)/2$ antisymmetric matrices are ordered first.

[1] M. Maniatis and O. Nachtmann, arXiv:1504.01736

Bilinear formalism

- ..In a way giving a lexicographic order of the doublets in the k first bilinears:

$$\{K_a\}_{a=1}^{k} = 2\{\widehat{C}_{12}, \widehat{C}_{13}, \ldots, \widehat{C}_{1N}, \widehat{C}_{23}, \ldots, \widehat{C}_{2N}, \widehat{C}_{34}, \ldots \\ \ldots\ldots, \widehat{C}_{N-1,N}\}, \tag{3}$$

where

$$\widehat{C}_{mn} \equiv \mathrm{Im}(\Phi_m^\dagger \Phi_n). \tag{4}$$

- The general NHDM potential V:

$$V = M_0 K_0 + M_a K_a + \Lambda_0 K_0^2 + L_a K_0 K_a + \Lambda_{ab} K_a K_b, \tag{1}$$

- Under a Higgs basis shift $\Phi_i \to \Phi_i' = U_{ij}\Phi_j$, $U \in SU(N)$, V transforms as

$$\begin{aligned} M_0 &\to M_0, \quad \Lambda_0 \to \Lambda_0, \\ M &\to R(U)M, \quad L \to R(U)L, \\ \Lambda &\to \Lambda' = R(U)\Lambda R^T(U), \end{aligned} \tag{5}$$

- where $R(U) \in \mathrm{Ad}_{SU(N)} \subset SO(N^2-1)$ is given by

$$U^\dagger \lambda_a U = R_{ab}(U)\lambda_b. \tag{6}$$

- M and L are examples of "adjoint vectors" since they transform as vectors under the adjoint representation $\mathrm{Ad}_{SU(N)}$.
- Λ also consists of adjoint vectors (its eigenvectors) through its spectral decomposition (eigensystem expansion).

Adjoint vectors and F-product

- Adjoint vectors are connected to the Lie algebra $\mathfrak{su}(N)$ through the map

$$\begin{aligned} \Omega : \mathbb{R}^{N^2-1} &\to \mathfrak{su}(N) \\ a &\mapsto a_i \lambda_i . \end{aligned} \tag{7}$$

- Ω is an Lie algebra isomorphism when $\mathbb{R}^{N^2-1}$ is equipped with the F-product[2]

$$\begin{aligned} F : \mathbb{R}^{N^2-1} \times \mathbb{R}^{N^2-1} &\to \mathbb{R}^{N^2-1} \\ (a, b) &\mapsto f_{ijk} a_i b_j \equiv F_k^{(a,b)} \end{aligned} \tag{8}$$

where f_{ijk} are the structure constants of $\mathfrak{su}(N)$ (in GM basis).

[2] I. de Medeiros Varzielas and I. P. Ivanov, arXiv:1903.11110

F-product

- Let $X \equiv \Omega(x) = x_i \lambda_i$, then

$$F^{(a,b)} = c \quad \Longleftrightarrow \quad [A, B] = 2iC. \tag{9}$$

- Hence, F-product relations are invariant under Higgs basis shifts,

$$F^{(a,b)} = c \Longleftrightarrow F^{(a',b')} = c', \tag{10}$$

where $x' = R(U)x$.

CP2

- A potential has an order-2 CP (CP2) if and only if it has a real basis[3] $\Rightarrow$ a basis were Λ is of the block diagonal form

$$\Lambda = \begin{pmatrix} C_N & \mathbf{0} \\ \mathbf{0} & A_N \end{pmatrix}, \tag{11}$$

  where C_N and A_N are arbitrary real and symmetric $k \times k$ and $(N^2 - 1 - k) \times (N^2 - 1 - k)$ matrices, with $k \equiv N(N-1)/2$.
- Means that the k eigenvectors t_a corresponding to C_N generates the def. rep. of $\mathfrak{so}(N)$ through

$$\text{span}\{(t_a)_b \lambda_b\}_{a=1}^{k} = \text{span}(\lambda_1, \dots, \lambda_k) = \mathfrak{so}(N), \tag{12}$$

  since $(t_a)_b = 0$ for $b > k$, where λ_b are the generalized GM-matrices.

[3] J. F. Gunion and H. E. Haber, arXiv:0506227

CP2

- The def. rep. of $\mathfrak{so}(N)$ here is conserved by Higgs basis shifts:
- I.e. a Higgs basis shift $U \in SU(N) \Rightarrow$

$$t_a \to v_a = R(U) t_a \tag{13}$$

$\Rightarrow$ span$\{V_a\}_{a=1}^{k}$ is equivalent to the def. rep. of $\mathfrak{so}(N)$.

- Moreover, real basis $\Rightarrow$

$$L \cdot t_a = M \cdot t_a = 0 \quad \forall a \leq k \tag{14}$$

since k first elements of L and M are inducing complex parameters in V.

Main result, CP2

A NHDM potential is CP2-symmetric if and only if[4]

1. $k = \frac{N(N-1)}{2}$ of Λ's eigenvectors, $\{v_a\}_{a=1}^{k}$, form a basis for the defining representation of $\mathfrak{so}(N)$
2. $L \cdot v_a = M \cdot v_a = 0, \quad \forall a \in \{1, \ldots, k\}$ ("LM-orthogonality").

- The two conditions can be checked in any Higgs basis:

(a) First by checking we have at least k LM-orthogonal eigenvectors,

(b) then (if necessary) check if k LM-orthogonal eigenvectors form an algebra,

(c) check if the algebra is $\mathfrak{so}(N)$,

(d) check if it is the defining representation **N** of $\mathfrak{so}(N)$.

[4] $N = 3$ was solved in arXiv:0605153 by C. C. Nishi

Do k LM-orth. eigenvectors give an algebra?

- Set of $k = \frac{N(N-1)}{2}$ eigenvectors of Λ closes under F-product (i.e. commutator) $\Rightarrow$ algebra.
- One can in most cases avoid to blindly check the closure of all (max) $\binom{N^2-1}{k} \sim \frac{2^{N^2}}{\sqrt{2e\pi N}}$ sets of k eigenvectors:
- Done by calculating the structure constants Z_{abc} of $\mathfrak{su}(N)$ in the basis given by the eigenvectors of Λ.
- Z must be sparse for k eigenvecs that generate a subalgebra $\Rightarrow$ we can usually dismiss a lot of candidate eigenvecs at "a glance".
- Eigenvalue degeneracies $\Rightarrow$ any linear combination from each eigenspace must be checked $\Rightarrow$ Numerical methods similar to CS?

Which algebra, $\mathfrak{so}(N)$?

- In case we have a k dimensional algebra, is it $\mathfrak{so}(N)$?
- If N is even and the rank $r = N/2 \leq 11$, then $\mathfrak{so}(N)$ is the only possible algebra (cf. subalgebra tables).
- If N is odd and $r = (N-1)/2$ we have to calculate the root system to check if we have $\mathfrak{so}(N)$ (=linear algebra).

Which representation of $\mathfrak{so}(N)$?

- The defining rep. **N** of $\mathfrak{so}(N)$ is the only N-dimensional faithful rep. of $\mathfrak{so}(N)$ in $\mathfrak{su}(N)$, with only some low N exceptions:

Dimension	Representation
$N = 3$	$\mathbf{2} + \mathbf{1}$
$N = 4$	$\mathbf{2} + \mathbf{2}'$
$N = 5$	$\mathbf{4} + \mathbf{1}$
$N = 6$	$\mathbf{4} + \mathbf{1} + \mathbf{1}$ $\overline{\mathbf{4}} + \mathbf{1} + \mathbf{1}$
$N = 8$	$\mathbf{8_s}$ $\mathbf{8_c}$

- So if $N \neq 3, 4, 5, 6, 8$ we can conclude we have **N** and hence CP2 symmetry, otherwise, we have to calculate the highest weight (=linear algebra).

Canonical custodial symmetry

- Custodial $SO(4)_C \simeq (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathbb{Z}_2 \supset \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry protects the ρ parameter

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \vartheta_W} \approx 1 \qquad (15)$$

  from large radiative corrections.
- A symmetry of the SM potential, but not necessarily of the NHDM potential.

Canonical custodial symmetry

- Canonical custodial symmetry (CS) implies[5] identical $SU(2)_R$ action on all bidoublets in some doublet basis:

$$\begin{pmatrix} i\sigma_2\phi_i^* & \phi_i \end{pmatrix} \equiv B_{ii} \to U_L B_{ii} U_R^\dagger \,, \quad \forall\, i \in \{1, \ldots, N\}, \tag{16}$$

- Some cases of non-canonical custodial symmetries are possible, through non-uniform $SU(2)_R$ action.[6]

[5] C. C. Nishi, arXiv:1103.0252

[6] A. Pilaftsis, arXiv:1109.3787; N. Darvishi, A. Pilaftsis, arXiv:1912.00887.

Manifest CS

- CS potentials may be transformed to a characteristic block-diagonal form, similarly to CP2-symmetric potentials, were CS is manifest:

$$\Lambda_C = \begin{pmatrix} C_N & \mathbf{0} \\ \mathbf{0} & A_N \end{pmatrix} \tag{17}$$

- The "custodial block" C_N (a $k \times k$ matrix, $k = \frac{N(N-1)}{2}$) was arbitrary in the case of manifest CP2, but is severely restricted in case of manifest CS.
- C_N generated by CS terms of the form[7]

$$\lambda_{abcd} I^{(4)}_{abcd} = \lambda_{abcd} \left(\widehat{C}_{ab} \widehat{C}_{cd} + \widehat{C}_{ad} \widehat{C}_{bc} + \widehat{C}_{ac} \widehat{C}_{db} \right), \tag{18}$$

with $\widehat{C}_{ij} \equiv \mathrm{Im}(\Phi_i^\dagger \Phi_j)$

[7] C. C. Nishi, arXiv:1103.0252

Main result, CS

- Let $N > 2$ and $N \neq 8$. Then a potential V is custodial-symmetric $\iff$
- Λ has $k = N(N-1)/2$ LM-orthogonal normalized eigenvectors v_a, with the same eigenvalues and F-product relations as the normalized eigenvectors t_a of some instance of the custodial block C_N.
- Only $\Rightarrow$ holds if $N = 8$:
- Because "triality" yields 2 additional representations of $\mathfrak{so}(8)$, with the same F-product relations as the defining rep. of $\mathfrak{so}(8)$.
- We apply this result to get computable conditions for CS for $N = 3, 4$ and 5:[8]

[8] $N = 3$ already solved in arXiv:1103.0252 by C. C. Nishi

Canonical custodial symmetry, $N = 3$

- Custodial block

$$C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

- With simple, corresponding eigenvecs and eigenvalues

$$t_{ai} = \delta_{ai}, \quad \beta_a = 0, \quad a = 1, 2, 3. \tag{20}$$

- These normalized eigenvectors satisfy the F-product relations

$$2F^{(t_a, t_b)} = \epsilon_{abc} t_c \tag{21}$$

Canonical custodial symmetry, $N = 3$

- $\Leftrightarrow$ the associated matrices $T_d \equiv (t_d)_e \lambda_e$ yield the defining rep. , i.e. the **3**, of $\mathfrak{so}(3)$:

$$[T_a, T_b] = i\epsilon_{abc} T_c \tag{22}$$

- Note that the $\mathbf{2} + \mathbf{1}$ of $\mathfrak{so}(3)$ would have given a prefactor 1 instead of 2 in (21),[9] i.e.

$$1 \cdot F^{(t_a, t_b)} = \epsilon_{abc} t_c \tag{23}$$

[9] I. de Medeiros Varzielas and I. P. Ivanov, arXiv:1903.11110

Canonical custodial symmetry, $N = 4$

- Custodial block, $\alpha \in \mathbb{R}$:

$$C_4 = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

Canonical custodial symmetry, $N = 4$

- C_4 has eigenvectors with eigenvalues $\pm\alpha$

$$
\begin{aligned}
t_1^+ &= \frac{1}{\sqrt{2}}(+1, 0, 0, 0, 0, -1, \mathbf{0}_9)^T \\
t_2^+ &= \frac{1}{\sqrt{2}}(0, +1, 0, 0, +1, 0, \mathbf{0}_9)^T \\
t_3^+ &= \frac{1}{\sqrt{2}}(0, 0, -1, +1, 0, 0, \mathbf{0}_9)^T \\
t_1^- &= \frac{1}{\sqrt{2}}(+1, 0, 0, 0, 0, +1, \mathbf{0}_9)^T \\
t_2^- &= \frac{1}{\sqrt{2}}(0, +1, 0, 0, -1, 0, \mathbf{0}_9)^T \\
t_3^- &= \frac{1}{\sqrt{2}}(0, 0, +1, +1, 0, 0, \mathbf{0}_9)^T
\end{aligned}
\tag{25}
$$

Canonical custodial symmetry, $N = 4$

- And one finds that they satisfy the $\mathfrak{so}(4) \cong \mathfrak{so}(3)_\alpha \oplus \mathfrak{so}(3)_{-\alpha}$ F-product relations

$$\begin{aligned} \sqrt{2} F^{(t_a^\pm, t_b^\pm)} &= \epsilon_{abc} t_c^\pm \\ F^{(t_a^\pm, t_b^\mp)} &= 0 \end{aligned} \tag{26}$$

- We can apply this to check if an arbitrary 4HDM potential has CS.
- (26) is independent of bases of the $\mathfrak{so}(3)$'s, also when CS is not manifest.

Extended eigenvalue degeneracies, $N = 4$

- In case of extended degeneracies, i.e. 4 or more eigenvecs with eigenvalue $\pm\alpha$, numerical methods have to be applied.
- We suggest a method based on optimization which quickly solves even the most extreme degeneracies.
- In case $\alpha = 0$, the methods from our CP2-article may be applied to detect the def. repr. of $\mathfrak{so}(4)$.

Canonical custodial symmetry, $N = 5$

- $\binom{5}{4} = 5$ free parameters in $C_5 \Rightarrow$ detection of CS more difficult as the eigenvectors of C_5 are not constant, in contrast to $N = 3, 4$.
- However, we show C_5 always can be transformed to

$$C_5 = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

  by a rotation of the doublets.
- $\Rightarrow$ all instances of CS for the 5HDM are equivalent to (27).

Canonical custodial symmetry, $N = 5$

- 3+3 eigenvectors of C_5 corr. to eigenvalues $\pm\alpha$ satisfy $\mathfrak{so}(4) \cong \mathfrak{so}(3)_\alpha \oplus \mathfrak{so}(3)_{-\alpha}$ F-products

$$\begin{aligned} \sqrt{2} F^{(t_a^\pm, t_b^\pm)} &= \epsilon_{abc} t_c^\pm \\ F^{(t_a^\pm, t_b^\mp)} &= 0. \end{aligned} \tag{28}$$

- And together with the 4 nullvectors of C_5, the eigenvecs of C_5 generate the def. repr. of $\mathfrak{so}(5)$.
- F-products involving nullvecs will depend on chosen basis of the nullvecs.

Canonical custodial symmetry, $N = 5$

- May check if 10 candidate eigenvecs generate a subalgebra (closes under F-products) by e.g. applying projectors.
- Subalgebra tables show the only 10d subalgebra of $\mathfrak{su}(5)$ containing $\mathfrak{so}(4)$ is $\mathfrak{so}(5)$.
- Finally, the prefactor $\sqrt{2}$ in (28) ensures you have the def. repr. of $\mathfrak{so}(5)$.
- Hence, the incomplete F-product relations (28) are sufficient to establish CS for $N = 5$.

Extended eigenvalue degeneracies, $N = 5$

- Extended degeneracies, including the case $\alpha = 0$, may be handled similarly as for $N = 4$.
- We were able to solve the worst degeneracy, for completely generic numerical potentials, by numerically optimizing quartic polynomials of up to 90 variables (takes less than a few minutes on an ordinary desktop computer).
- For $N > 5$ the eigenvalue pattern essentially fades away, which makes it more difficult to decide whether or not a potential has a CS.