Unitarity constraints on large multiplets of arbitrary gauge groups

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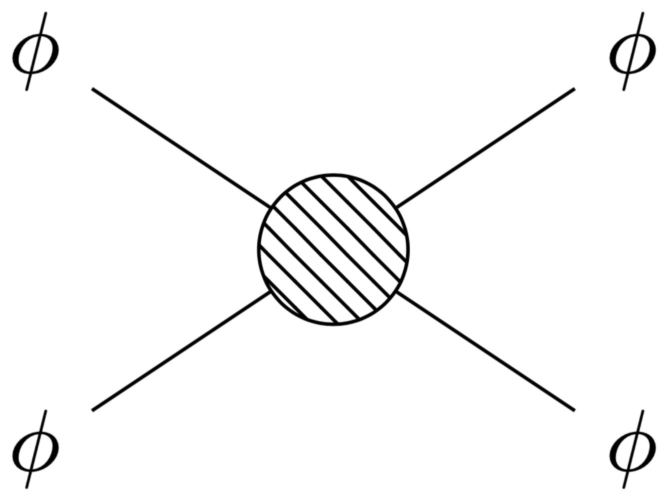
Preamble

Q: Can the dimension and number of scalar & fermionic multiplets be arbitrarily large?

Impose Partial Wave Unitarity → Pairs of fermion/scalar annihilating into gauge bosons

Partial Wave Unitarity Bounds

- Decompose scattering amplitude into partial waves:



$$= \mathcal{M}(\cos\theta) = 16\pi \sum_{J=0}^{\infty} a_J (2J+1) P_J(\cos\theta)$$

- Optical Theorem: scattering amplitudes cannot grow arbitrarily large

$$|a_J| \leq 1$$

$$0 \leq \text{Im}\{a_J\} \leq 1$$

$$|\text{Re}\{a_J\}| \leq \tfrac{1}{2}$$

Strongest Constraint →

$$|\text{Re}\{a_0\}| \leq \frac{1}{2}$$

One multiplet & one gauge group

- Consider theory symmetric under **one** gauge group $\boldsymbol{G}$
- Contain **one** fermionic OR scalar multiplet of dimension $\boldsymbol{n}$
- Working in unmixed basis and high-energy limit:

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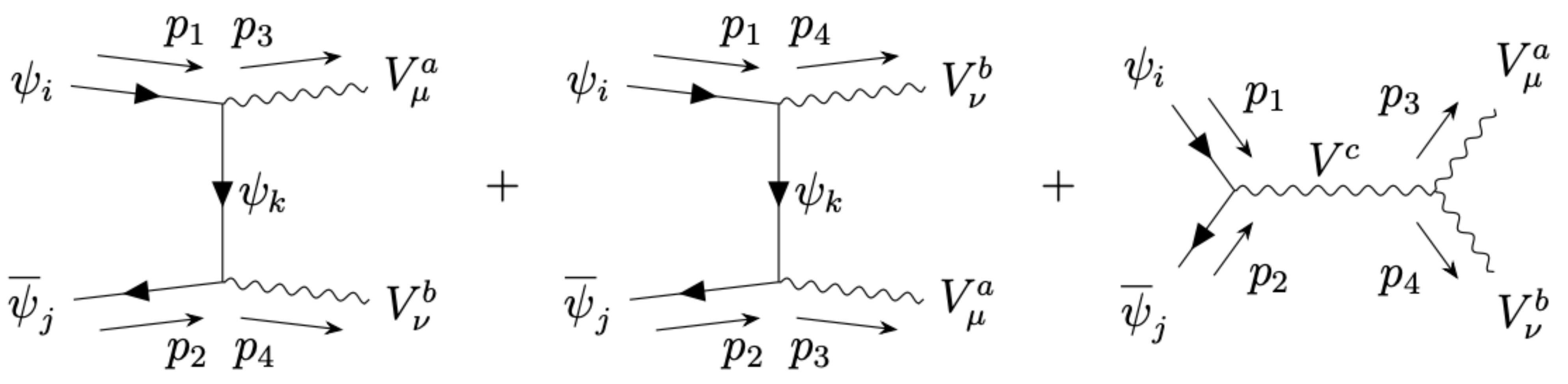

$$a_0 \sim g^2 \left(3T_aT_b + T_bT_a\right)_{ij}$$

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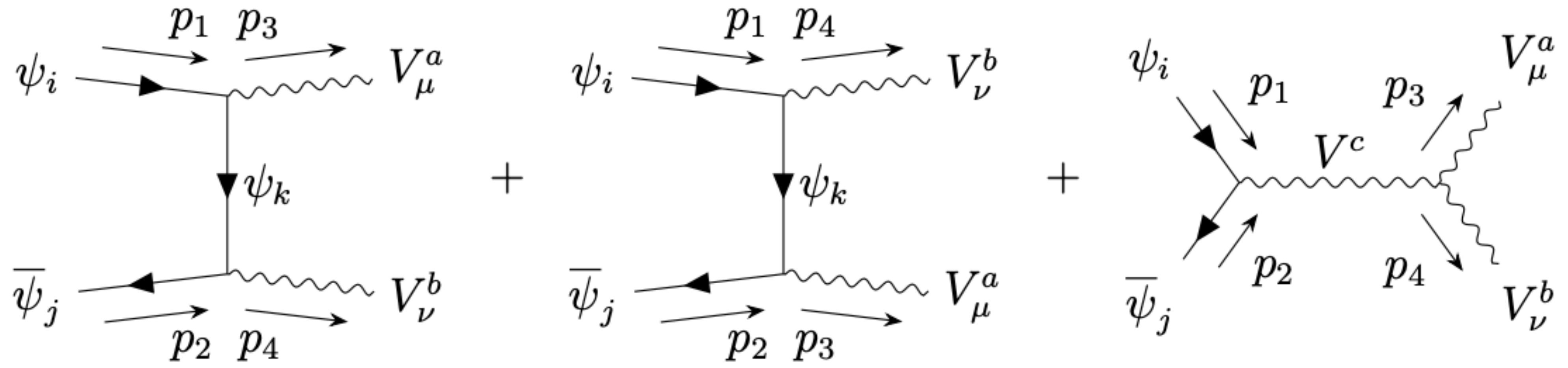

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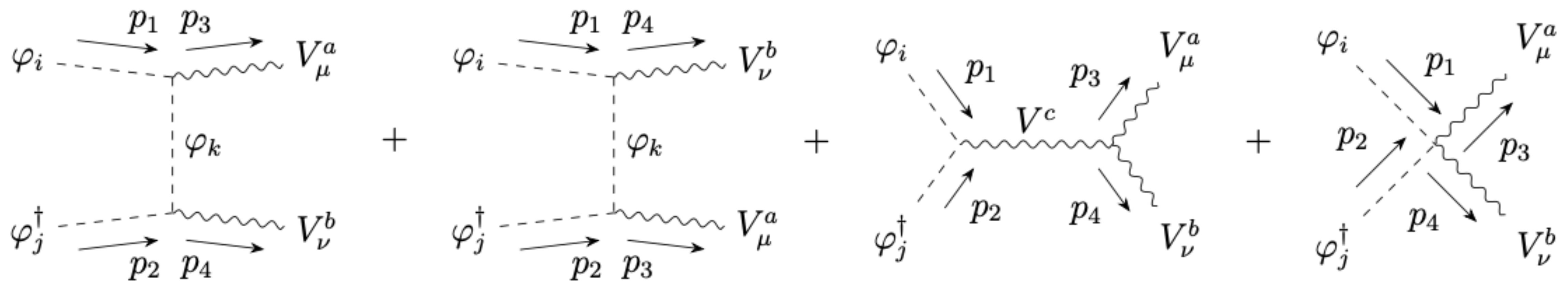

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Gauge Coupling

Multiplet Dimension

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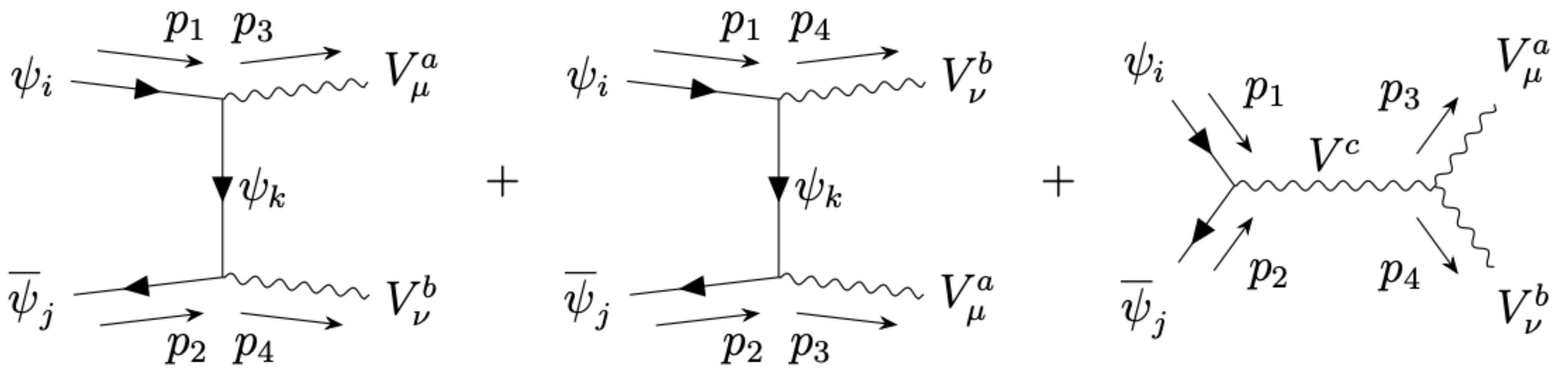

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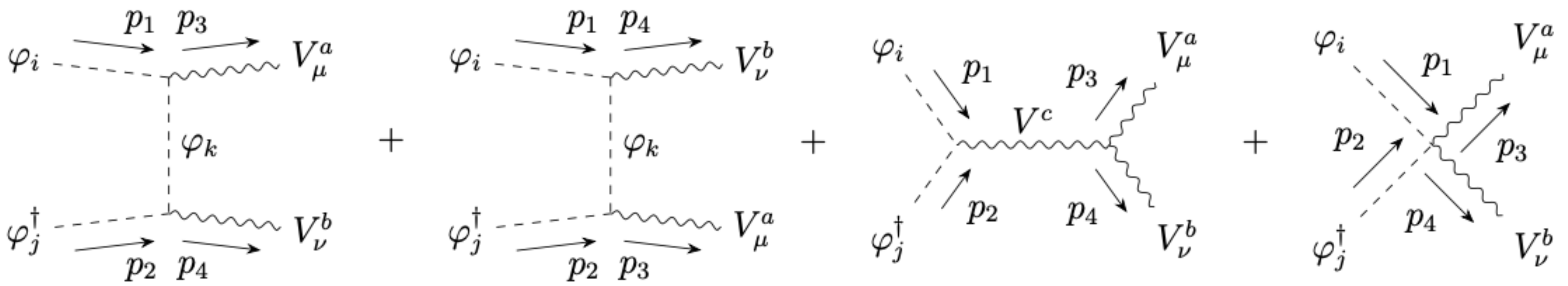

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Coupled channel analysis

- There are many states (with same quantum numbers) scattering into $|VV\rangle$
- Partial unitarity bounds must also apply to superpositions of states with same quantum numbers!

$$\left\langle V^a V^b \left| a_0 \right| \psi_i \overline{\psi}_j \right\rangle \leq \frac{1}{2} \quad \longleftrightarrow \quad \sum_{ijab} k_{ijab} \left\langle V^a V^b \left| a_0 \right| \psi_i \overline{\psi}_j \right\rangle \leq \frac{1}{2}$$

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- Largest a_0 arise from symmetrical (*i.e.* group invariant) superposition of states

$$|\boldsymbol{n}_\psi \boldsymbol{n}_\psi\rangle_{\text{sym}} \equiv \frac{1}{\sqrt{n}} \sum_i |\psi_i \overline{\psi}_i\rangle \qquad |\boldsymbol{n}_\varphi \boldsymbol{n}_\varphi\rangle_{\text{sym}} \equiv \frac{1}{\sqrt{n}} \sum_i |\varphi_i \varphi_i^\dagger\rangle \qquad |VV\rangle_{\text{sym}} \equiv \frac{1}{\sqrt{2\tilde{n}}} \sum_a |V^a V^a\rangle$$

Initial States — Final State

Coupled channel analysis

- Symmetric states allow for simplifications using group theory identities:

Casimir Invariant

Fermion Multiplet: $$|a_0^\psi| = \frac{g^2}{16}\sqrt{\frac{n}{\widetilde{n}}}\,C(n)$$

Scalar Multiplet: $$|a_0^\varphi| = \frac{g^2}{8\pi}\sqrt{\frac{n}{\widetilde{n}}}\,C(n)$$

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Multiplet dimension cannot be arbitrarily large!!!

Multiple multiplets & multiple gauge groups

- Now consider theory with:
  - $\boldsymbol{N_F}$ fermionic multiplets with dimension $\boldsymbol{n_{Fi}}$
  - $\boldsymbol{N_S}$ scalar multiplets with dimension $\boldsymbol{n_{Si}}$
  - Symmetric under $G_1 \times G_2 \times G_3 \times \ldots$
- Using coupled channel analysis, largest a_0 is the largest eigenvalue of:

$$\begin{pmatrix} |a_0^{F_1}|_{G_1} & |a_0^{F_2}|_{G_1} & \cdots & |a_0^{S_1}|_{G_1} & |a_0^{S_2}|_{G_1} & \cdots \\ |a_0^{F_1}|_{G_2} & |a_0^{F_2}|_{G_2} & \cdots & |a_0^{S_1}|_{G_2} & |a_0^{S_2}|_{G_2} & \cdots \\ |a_0^{F_1}|_{G_3} & |a_0^{F_2}|_{G_3} & \cdots & |a_0^{S_1}|_{G_3} & |a_0^{S_2}|_{G_3} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$$

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Single Gauge Group

$$|a_0| = \sqrt{\sum_{i=1}^{N_F}\left[a_0^{F_i}\right]^2 + \sum_{i=1}^{N_S}\left[a_0^{S_i}\right]^2}$$

$$a_0^{F_i} = \frac{g^2}{16}\sqrt{\frac{n}{\tilde{n}}}\,C(\boldsymbol{n}_{F_i}) \qquad a_0^{S_i} = \frac{g^2}{8\pi}\sqrt{\frac{n}{\tilde{n}}}\,C(\boldsymbol{n}_{S_i})$$

Grand Unified Theory $\boldsymbol{E_6}$

- Fermions are in three families of **27** – dimensional multiplets

- Correct fermion mass hierarchy is only achieved with a **351'** scalar multiplet

- Using conservative estimate for $\boldsymbol{g}$ at GUT scale $\Rightarrow$ Violation of unitarity

- More scalar multiplets are needed to break $E_6 \rightarrow$ SM

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Raise questions about viability of
any perturbative $\boldsymbol{E_6}$ GUT!

Multi-Higgs Models

- Consider a universe with N_{Higgs} doublets
- $N_{\text{Higgs}} = 1$: $|a_0| \approx 10^{-2}$
- In general: $|a_0| \approx 10^{-2}\sqrt{N_{Higgs}}$

$$|a_0| \leq \frac{1}{2} \Leftrightarrow N_{Higgs} \leq 2262$$

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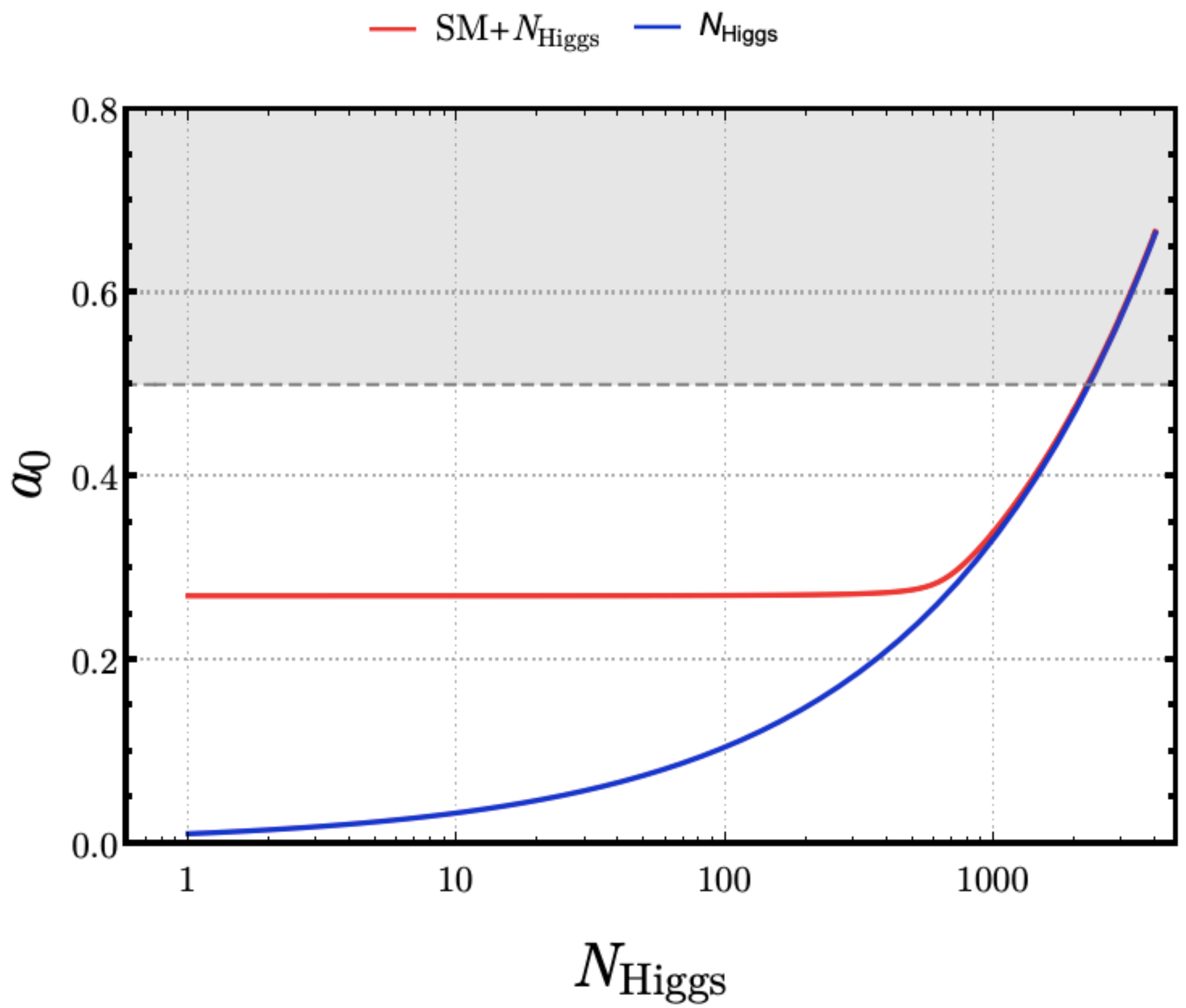
- Full calculation with SM+N_{Higgs} asymptotically approaches $\propto \sqrt{N_{Higgs}}$



Final Remarks

- Imposed Partial Wave Unitarity Bounds

  ⇒ Dimension of multiplets cannot be arbitrarily large

  ⇒ Number of multiplets cannot be arbitrarily large

- Powerful formulas to quickly check validity of perturbation theory

- All E_6 scenarios violate perturbative unitarity

- In NHDM, "N" cannot be arbitrarily large if we want theory to be perturbative

Backup Slides

Scattering Matrix

$$a_0 = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} |a_0^{F_1}|_{G_1} & |a_0^{F_2}|_{G_1} & \cdots & |a_0^{S_1}|_{G_1} & |a_0^{S_2}|_{G_1} & \cdots \\ |a_0^{F_1}|_{G_2} & |a_0^{F_2}|_{G_2} & \cdots & |a_0^{S_1}|_{G_2} & |a_0^{S_2}|_{G_2} & \cdots \\ |a_0^{F_1}|_{G_3} & |a_0^{F_2}|_{G_3} & \cdots & |a_0^{S_1}|_{G_3} & |a_0^{S_2}|_{G_3} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$$

Constraints on SM extension

- SM particle content is well compatible with unitarity: $|a_0| \approx 0.27$

$$\psi \sim \left(\boldsymbol{n}_\psi^{SU(3)},\, \boldsymbol{n}_\psi^{SU(2)},\, Y_\psi\right)$$

$\boldsymbol{n}_\psi^{SU(2)}$ \ $\boldsymbol{n}_\psi^{SU(3)}$: $\lvert Y_\psi\rvert^{\max}$	**1**	**3**	**6**	**8**
1	7.91	5.96	4.41	4.04
2	6.65	4.96	2.65	2.13
3	6.00	4.39	—	—
4	5.52	3.87	—	—
5	5.05	2.95	—	—
6	4.37	—	—	—
7	2.13	—	—	—

$$\varphi \sim \left(\boldsymbol{n}_\varphi^{SU(3)},\, \boldsymbol{n}_\varphi^{SU(2)},\, Y_\varphi\right)$$

$\boldsymbol{n}_\varphi^{SU(2)}$ \ $\boldsymbol{n}_\varphi^{SU(3)}$: $\lvert Y_\varphi\rvert^{\max}$	**1**	**3**	**6**	**8**	**10**
1	9.92	7.51	6.04	5.60	3.03
2	8.34	6.29	4.78	4.40	—
3	7.53	5.64	3.94	3.56	—
4	6.97	5.15	2.86	2.10	—
5	6.51	4.62	—	—	—
6	6.03	3.74	—	—	—
7	5.38	—	—	—	—
8	4.08	—	—	—	—