Symmetry options for multi-Higgs-doublet models

Igor Ivanov

School of Physics and Astronomy, SYSU, Zhuhai

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Based on: J. Shao, I.P. Ivanov, JHEP 04 (2023) 116
J. Shao, I.P. Ivanov, M. Korhonen, 2404.10349 = J. Phys. A xx (2024) xxx
and earlier papers

中山大學 物理与天文学院
SUN YAT-SEN UNIVERSITY SCHOOL OF PHYSICS AND ASTRONOMY

Ilya Ginzburg (1934–2024)

Ilya Ginzburg and Maria Krawczyk at Multi-Higgs 2012

Symmetries in multi-Higgs-doublet models

NHDM vs 2HDM

Higgses can come in generations → N-Higgs-doublet models (NHDMs)

2HDMs explored in thousands of papers. What new can NHDMs with $N > 2$ bring?

- More options for model-building (scalar and fermion) ⇒ richer pheno!
  - new options for CP violation [Branco, Gerard, Grimus, 1984] and exotic CP symmetries;
  - possible insights into the flavor puzzle;
  - combining features of 2HDM: NFC + CPV [Weinberg, 1976; Branco, 1979], scalar DM + CPV [Grzadkowski et al, 2009].
  - astroparticle consequences: various dark sectors [Cordero et al, 2017]; new options for baryon asymmetry [Davoudiasl, Lewis, Sullivan, 2019]; multi-step phase transitions etc.

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  - astroparticle consequences: various dark sectors [Cordero et al, 2017]; new options for baryon asymmetry [Davoudiasl, Lewis, Sullivan, 2019]; multi-step phase transitions etc.
- Many of these features are driven by symmetries, exact or approximate [many classic papers];

The wealth of symmetry options is the single most powerful novelty of the NHDMs.

Symmetries as anchor structures

The general NHDM potential has $N^2(N^2+3)/2$ free parameters, many more in the Yukawa sector.

Symmetry-based NHDMs are anchor structures in the vast parameters space.

They guide your exploration of the NHDM pheno richness.

A possible strategy in the ideal world:

- identify all symmetry options (= all groups, all irrep assignments, all ways symmetries can be broken) available in the NHDM, for the scalar and Yukawa sectors;
- establish basis-independent methods for detection of each symmetry;
- compute basic pheno consequences for each symmetry option;
- explore parameter space in the vicinity of each symmetric NHDM, either with soft or hard breaking terms.

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Constructive methods needed for the full classification of symmetry groups!

In the 2HDM, the (short) list of symmetry groups is easily established via geometric constructions in the bilinear space developed in [Nachtmann et al, 2004–2007; Ivanov, 2006–2007; Nishi, 2006–2008].

Can be generalized to the 3HDM [Ivanov, Nishi, 1004.1799; Maniatis, Nachtmann, 1408.6833], but it does not offer an easy shortcut to the list of symmetry groups.

Symmetries in the 3HDM

Which symmetry groups G are available for the 3HDM scalar sector?

Constructive methods developed and used in 2011-2013:

- rephasing groups via the Smith normal form technique [Ivanov, Keus, Vdovin, 1112.1660]

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad U(1), \quad U(1) \times \mathbb{Z}_2, \quad U(1) \times U(1) .$$

- finite non-abelian groups with some help of finite group theory [Ivanov, Vdovin, 1210.6553]

$$S_3, \quad D_4, \quad A_4, \quad S_4, \quad \Delta(54), \quad \Sigma(36) .$$

These lists are exhaustive.

- Continuous symmetry groups of the potential alone, relying on the extra possibility of intra-doublet transformations; the full classification in Darvishi, Pilaftsis, 1912.00887. Re-derived the above list of finite non-abelian groups.

Finite non-abelian groups for the 3HDM scalar sector

Building finite non-abelian G from the abelian building blocks by group extension:

- G can only have the following abelian subgroups: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.
- Their orders involve only two prime factors: 2 and 3. Therefore, $|G| = 2^a 3^b$.
- Burnside's $p^a q^b$ theorem + some finite group theory $\Rightarrow$ G contains a normal maximal abelian subgroup A.
- $\Rightarrow$ the group G can only be of the type $G = A \rtimes K$, where $K \subseteq Aut(A)$.
- The full classification reduces to explicit checks of very few cases [Ivanov, Vdovin, 1210.6553].

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A take-home message

Classification of symmetries in a given BSM model is not limited to guesses or blind scans.

Group theory may help you with unexpected insights.

In particular, identification of the rephasing symmetry group in any BSM model can be done algorithmically with the aid of the Smith normal form.

Symmetries in 4HDM

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We found 50+ papers based on 4HDM equipped with global symmetries.

- The earliest 4HDM is Bjorken, Weinberg, PRL38, 622 (1977), possible muon number non-conservation.
- 1980's: G-based 3HDMs sometimes complemented with the fourth doublet transforming as a singlet under G.
- Ma, PRD43, 2761 (1991) + follow-up papers: 4HDM with $G = S_3 \times \mathbb{Z}_3$ and irrep assignment $1 + 1' + 2$; economic description of fermion properties, including the 17 keV neutrino signal.

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- NFC in 4HDM: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with various sign assignments for fermions; e.g. Gonçalves, Knauss, Sher, 2301.08641: Φ_q couples to quarks, Φ_e, Φ_μ, Φ_τ couple to e, μ, τ $\to$ FCNC for leptons, not for quark.
- "Duplication" of the 2HDM, e.g. supersymmetrization of the 2HDM, Twin Higgs models, etc.

Our strategy

No one has studied all the symmetry options in 4HDM systematically.

We decided to give it a try (constructively!) for finite symmetry groups.

- We used the same approach that worked in the 3HDM: start with finite abelian groups and construct non-abelian ones via group extension.
- Unlike in the 3HDM, there is no guarantee that this procedure exhausts all possible cases in the 4HDM. Still, it gives all groups of reflections + rephasings, which are available for the 4HDM.

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Finite abelian groups in the 4HDM

An important technical subtlety:

- We need abelian subgroups not of $SU(4)$, but of $PSU(4) \simeq SU(4)/\mathbb{Z}_4$, where $\mathbb{Z}_4$ is the center of $SU(4)$ and is generated by diag(i, i, i, i).

The recipe for finite abelian symmetry groups given in [Ivanov, Keus, Vdovin, 1112.1660].

- Rephasing groups (abelian in $SU(4)$ and in $PSU(4)$): all abelian groups A of order $|A| \leq 8$:

$$A = \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_5, \quad \mathbb{Z}_6, \quad \mathbb{Z}_7, \quad \mathbb{Z}_8 \quad \Rightarrow \text{ paper } 1 = 2305.05207$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \quad \Rightarrow \text{ paper } 2 = 2404.10349.$$

- Beyond rephasing: abelian subgroups of $PSU(4)$ whose pre-images in $SU(4)$ are non-abelian:

$$A = \mathbb{Z}_4 \times \mathbb{Z}_4\,, \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2\,, \qquad (\mathbb{Z}_2)^4\,.$$

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Extensions by cyclic groups

Paper 1: Shao, Ivanov, JHEP 04 (2023) 116 = 2305.05207, extensions by cyclic groups.

The procedure:

- Pick up an abelian group $A = \mathbb{Z}_q$, $q \leq 8$.
- Find its automorphism group $\mathrm{Aut}(A)$ and list all of its subgroups $K \subseteq \mathrm{Aut}(A)$.
- For each K, non-abelian extensions of K by A are either
  - semidirect products $G = A \rtimes K$, or
  - non-split extensions $G = A\,.\,K$,

  defined by the action of K on A and by closure of K inside G.
- Check explicitly whether the group fits the 4HDM.

A	$\mathrm{Aut}(A)$
$\mathbb{Z}_2$	$\{e\}$
$\mathbb{Z}_3$	$\mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_2$
$\mathbb{Z}_5$	$\mathbb{Z}_4$
$\mathbb{Z}_6$	$\mathbb{Z}_2$
$\mathbb{Z}_7$	$\mathbb{Z}_6$
$\mathbb{Z}_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

An example: extensions by $\mathbb{Z}_7$

The potential of the $\mathbb{Z}_7$-symmetric 4HDM contains the generic rephasing-insensitive part

$$V_0 \;=\; \sum_{i=1}^{4}\left[m_{ii}^2(\phi_i^\dagger\phi_i)+\Lambda_{ii}(\phi_i^\dagger\phi_i)^2\right]+\sum_{i<j}\left[\Lambda_{ij}(\phi_i^\dagger\phi_i)(\phi_j^\dagger\phi_j)+\tilde{\Lambda}_{ij}(\phi_i^\dagger\phi_j)(\phi_j^\dagger\phi_i)\right]\,.$$

and the following rephasing-sensitive terms:

$$V(\mathbb{Z}_7)=\lambda_1(\phi_2^\dagger\phi_1)(\phi_4^\dagger\phi_1)+\lambda_2(\phi_3^\dagger\phi_2)(\phi_4^\dagger\phi_2)+\lambda_3(\phi_1^\dagger\phi_3)(\phi_4^\dagger\phi_3)+h.c.,$$

which are invariant under

$$a=\mathrm{diag}(\eta,\,\eta^2,\,\eta^4,\,1)\,,\quad \eta\equiv e^{2\pi i/7}\,,\quad \eta^7=1.$$

An example: extensions by $\mathbb{Z}_7$

Since $\mathrm{Aut}(\mathbb{Z}_7) = \mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$, we define two generators b (of order 2) and c (of order 3):

- b such that $b^2 = 1$ and $b^{-1}ab = a^{-1}$, leading to $\mathbb{Z}_7 \rtimes \mathbb{Z}_2 \simeq D_7$;
- c such that $c^3 = 1$ and $c^{-1}ac = a^2$, leading to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3 \simeq T_7$.

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- c such that $c^3 = 1$ and $c^{-1}ac = a^2$, leading to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3 \simeq T_7$.

It turns out that the first extension is not available in the 4HDM: the equation $b^{-1}ab = a^{-1}$, with the known a, does not have solutions in the 4D space. That is, $b \notin PSU(4)$.

Not all extensions which exist group-theoretically fit the model space.

An example: extensions by $\mathbb{Z}_7$

The second option, $c^3 = 1, c^{-1}ac = a^2$, has solutions in $PSU(4)$ and leads to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3 \simeq T_7$:

$$c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{cyclic permutation } \phi_1 \mapsto \phi_3 \mapsto \phi_2 \mapsto \phi_1 .$$

The rephasing-sensitive potential:

$$V(T_7) = \lambda \left[(\phi_2^\dagger \phi_1)(\phi_4^\dagger \phi_1) + (\phi_3^\dagger \phi_2)(\phi_4^\dagger \phi_2) + (\phi_1^\dagger \phi_3)(\phi_4^\dagger \phi_3) + h.c. \right],$$

and corresponding conditions on the rephasing-insensitive potential V_0.

Extensions by cyclic groups: results

Remarks:

- Not all extensions can fit the 4HDM space!

  $\mathbb{Z}_8$ cannot be extended to a finite group $\rightarrow$ accidental continuous symmetry.

- The same group can be implemented in several non-equivalent ways.

  There are three non-equivalent forms of $\mathbb{Z}_4$ in the 4HDM; their extensions also differ.

- Some groups automatically lead to CP conservation. Others allow for explicit or spontaneous CP violation.

A	extension	G	$\|G\|$
$\mathbb{Z}_2$	—	—	—
$\mathbb{Z}_3$	$\mathbb{Z}_3 \rtimes \mathbb{Z}_2$	S_3	6
$\mathbb{Z}_4$	$\mathbb{Z}_4 \rtimes \mathbb{Z}_2$	D_4	8
	$\mathbb{Z}_4 \,.\, \mathbb{Z}_2$	Q_4	8
$\mathbb{Z}_5$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$GA(1,5)$	20
	$\mathbb{Z}_5 \rtimes \mathbb{Z}_2$	D_5	10
$\mathbb{Z}_6$	$\mathbb{Z}_6 \rtimes \mathbb{Z}_2$	D_6	12
$\mathbb{Z}_7$	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	T_7	21
$\mathbb{Z}_8$	—	—	—

Extensions by products of cyclic groups

Paper 2: Shao, Ivanov, Korhonen, J. Phys. A (2024) = 2404.10349, extensions by products of cyclic groups.

- The extension procedure is the same: write $\mathrm{Aut}(A)$, list $K \subseteq \mathrm{Aut}(A)$, construct extensions $G = A \rtimes K$ or $G = A\,.\,K$.
- However, now $\mathrm{Aut}(A)$ are non-abelian and large.
- $\Rightarrow$ a lot of subgroups! For example, $GL(3,2)$ has 179 subgroups. Should we check them one by one??

A	$\mathrm{Aut}(A)$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	S_3
$\mathbb{Z}_2 \times \mathbb{Z}_4$	D_4
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$GL(3,2)$

Extensions of conjugate groups are isomorphic

Theorem: extensions based on conjugate subgroups are isomorphic.

If $K_1 \subset \mathrm{Aut}(A)$ and $K_2 \subset \mathrm{Aut}(A)$ are conjugate to each other, meaning there exists $q \in \mathrm{Aut}(A)$ such that $q^{-1}K_1 q = K_2$, then for any $G_1 = A\,.\,K_1$, there is an isomorphic extension $G_2 = A\,.\,K_2$.

This result hugely simplifies the analysis: we need to list not all subgroups of $\mathrm{Aut}(A)$ but all conjugacy classes of subgroups.

For $GL(3,2) = \mathrm{Aut}((\mathbb{Z}_2)^3)$, we have 13 non-equivalent conjugacy classes of subgroups:

Representative Subgroups	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	S_3	$\mathbb{Z}_7$	D_4	A_4	$\mathbb{Z}_3 \rtimes \mathbb{Z}_7$	S_4
Number of conjugacy classes	1	1	1	2	1	1	1	2	1	2

Moreover, subgroups which contain $\mathbb{Z}_7$ do not fit the 4HDM space and can be skipped.

Extensions by products of cyclic groups: results

A	extensions	G	$\|G\|$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$A \rtimes \mathbb{Z}_2$	D_4	8
	$A \rtimes \mathbb{Z}_3$	A_4	12
	$A \rtimes S_3$	S_4	24
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$A \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	16
	$A \rtimes \mathbb{Z}_2$	SmallGroup(16,3)	16
	$A \rtimes \mathbb{Z}_2$	SmallGroup(16,13)	16
	$A \rtimes \mathbb{Z}_4$	SmallGroup(32,6)	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	SmallGroup(32,49)	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	SmallGroup(32,27)	32
	$A \rtimes D_4$	$UT(4,2)$	64

A	extensions	G	$\|G\|$
$(\mathbb{Z}_2)^3$	$A \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	16
	$A\,.\,\mathbb{Z}_2$	SmallGroup(16,3)	16
	$A \rtimes \mathbb{Z}_3$	$\mathbb{Z}_2 \times A_4$	24
	$A \rtimes \mathbb{Z}_4$	SmallGroup(32,6)	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	SmallGroup(32,49)	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	SmallGroup(32,27)	32
	$A\,.\,(\mathbb{Z}_2 \times \mathbb{Z}_2)$	SmallGroup(32,34)	32
	$A \rtimes S_3$	$\mathbb{Z}_2 \times S_4$	48
	$A \rtimes D_4$	$UT(4,2)$	64
	$A \rtimes S_4$	SmallGroup(192,955)	192

Many of these groups are labeled by their GAP ids and do not possess any “simple” name.

The main result

- 3HDM: the full list of finite non-abelian groups was compact: S_3, D_4, A_4, S_4, $\Delta(54)$, $\Sigma(36)$.
- 4HDM: many non-abelian groups, some of them bearing dull labels and look unfamiliar to the BSM community.

The main result of this work is not these lists themselves but the methods which we developed and used to exhaustively list the cases.

If you wish to explore a symmetry-based 4HDM, use the methods, not the final answer.

Try the same constructive approach in other BSM models: first get the list of all abelian groups [Ivanov, Keus, Vdovin, 1112.1660] and apply the group extension technique.

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Conclusions

Understanding the full phenomenological richness of the NHDMs is unthinkable without knowing all symmetry-based options.

- In the 3HDM, the systematic classification is based on constructive group-theoretical methods, in particular on the full list of abelian groups and on the group extension technique.
- In 2305.05207, 2404.10349, we applied the same technique to the scalar sector of the 4HDM. As we encountered novel challenges, we had to further develop the technique.
- So far, we found all finite non-abelian groups that can be constructed as group extensions $A \rtimes K$ or $A\,.\,K$, where A is a rephasing group and K is a group of (some of) its automorphisms.
- Still, our lists do not exhaust all symmetry options for the 4HDM $\Rightarrow$ work remains to be done. My personal feeling is that there will be very few additional groups.