4.2 Deep inelastic scattering

So far we have dealt with elastic $eN$ scattering from where we can extract the nucleon’s electromagnetic form factors in the spacelike region. In a general inelastic scattering process the nucleon will break up and produce all possible hadronic final states. Depending on the invariant mass of the hadronic end product, the inelastic cross section will exhibit nucleon resonance peaks and contain nucleon-meson continua. Moreover, deep inelastic scattering probes the composite nature of the nucleon: it gives us access to the parton distribution functions which measure the longitudinal momentum distributions of quarks and gluons inside the nucleon.

**Phase space in inelastic scattering.** We continue to work with the variables defined in Eq. (D.4). In the limit of massless electrons ($k_i^2 = k_f^2 = 0$) we have now three independent Lorentz invariants: the spacelike momentum transfer $\tau \geq 0$, the inelasticity $\omega \geq 0$, and the crossing variable $\nu$. It is experimentally convenient to work with the variables $E$, $\theta$ and $W$, where $E$ is the energy of the incoming lepton in the lab frame, $\theta$ is the lepton scattering angle, and $W = M\sqrt{1 + 4\omega}$ is the invariant mass of the hadrons in the final state (see Eqs. (D.7) and (D.14)). In the one-photon exchange approximation, the cross section factorizes again in a leptonic and a hadronic part, where the hadronic subprocess depends only on $\tau$ and $\omega$.

The resulting phase space in the $(\omega, \tau)$ plane at fixed $E$ is sketched in Fig. 4.4. If we solve the relations (D.14) for $\tau(\omega, \theta, E)$ we obtain

$$\tau = (\varepsilon - \omega) \frac{4\varepsilon \sin^2 \theta}{1 + 4\varepsilon \sin^2 \theta}, \quad \varepsilon := \frac{E}{2M}. \quad (4.51)$$

For constant lepton energy $E$, the physically allowed region is then bounded by $\omega = 0$ (elastic scattering), $\theta = 0 \Rightarrow \tau = 0$ (forward angles), and backward angles $\theta = \pi \Rightarrow \tau = (\varepsilon - \omega) 4\varepsilon/(1 + 4\varepsilon) \approx \varepsilon - \omega$, which is the triangular blue area in the plot. Its size is characterized by the external control parameter $E$: if we increase the energy of the lepton beam we can reach higher $\tau$ and $\omega$ values. The invariant mass $W$ is directly related to the variable $\omega$: the elastic threshold $\omega = 0$ corresponds to $W = M$, and $0 < \omega \lesssim 1$ is the region where nucleon resonances appear at fixed $W$, starting with the $\Delta(1232)$ peak; see Fig. 4.5. Above $W \sim 2$ GeV, there is no visible resonance structure left in the cross section. The limit $\tau + \omega \to \infty$ and $\omega/\tau = const.$ defines the Bjorken limit: this is the region of deep inelastic scattering (DIS) where scaling occurs (more on that below). From Eqs. (D.7) and (D.8) we have

$$\omega + \tau = \frac{\nu'}{2M}, \quad 1 + \frac{\omega}{\tau} = \frac{1}{x}, \quad (4.52)$$

and hence the Bjorken limit means $\nu' \to \infty$ and constant Bjorken-$x$. The lines of constant $x$ are shown in Fig. 4.4; the limit $\omega = 0 \Leftrightarrow x = 1$ corresponds to elastic scattering.\footnote{It is more common to plot Fig. 4.4 in terms of $\nu'$ and $\tau$. The upper right quadrant in the $(\omega, \tau)$ plane is then squeezed in the $(\nu', \tau)$ plane; the elastic limit $\omega = 0$ becomes the line $\tau = \nu'/(2M)$.}
Cross section and structure functions. Let’s work out the cross section for inelastic $eN$ scattering. In an inclusive measurement only the outgoing electron is detected but not the remnants of the proton. The cross section in the one-photon approximation has still the generic form of Eqs. (4.18–4.19) with the same leptonic tensor (4.25); however, the hadronic contribution to the invariant matrix element $|M|^2$ and to the phase space factor now sums over all possible final states. We can absorb the integral over $d^3 p_f$ and the $\delta-$function for energy-momentum conservation into a generic hadronic tensor $W^{\mu\nu}$, whose explicit form we will discuss later in Eq. (4.79):

$$d\sigma = \frac{1}{4ME} \frac{d^3 k_f}{(2\pi)^3 2E'} \frac{e^4}{q^4} L_{\mu\nu} 4\pi M W^{\mu\nu} \Rightarrow \frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{q^4} \frac{E'}{E} L_{\mu\nu} W^{\mu\nu}. \quad (4.53)$$

Since current conservation must still hold because the sum of the outgoing charges must equal the nucleon charge, $W^{\mu\nu}$ must be transverse in its Lorentz indices. The most general transverse tensor that we can construct according to these constraints is given by

$$W^{\mu\nu} = -W_1(\tau, \omega) T^{\mu\nu}_q + \frac{W_2(\tau, \omega)}{M^2} p_T^{\mu} p_T^{\nu}, \quad (4.54)$$

$$\text{Figure 4.4:} \text{ Phase space in inelastic } eN \text{ scattering in the variables } \tau \text{ and } \omega \text{ at fixed lepton energy } E. \text{ The various regions illustrate: the physically allowed phase space (triangular, blue), the resonance region (brown) and the deep inelastic region (red). Lines of constant scattering angle } \theta \text{ and Bjorken-} x \text{ are also shown.}$$

\footnote{In principle there would be three more tensor structures apart from the two written here, namely $q^\nu q^\tau$ and $p_T^{\mu} q^\nu \pm q^\nu p_T^{\nu}$, but the transversality constraints $q^\nu W^{\mu\nu} = 0$, $W^{\mu\nu} q^\nu = 0$ ensure that their dressing functions vanish. Moreover, we only deal with unpolarized scattering; in general there would be two further spin-dependent structure functions $\sim g_1, g_2$ and also another structure in the lepton tensor which only contribute to polarized scattering.}
where the transverse projector and momenta were defined in Eq. (4.26). The two response functions $W_1$ and $W_2$ depend on the Lorentz invariants $\tau$ and $\omega$. Combining this with the leptonic tensor yields
\[
L^\mu\nu W_{\mu\nu} = 4 \left[ \frac{W_2}{M^2} \left( \frac{(p \cdot k)^2}{4} + \frac{q^2}{4} p_T^2 \right) - W_1 \left( k^2 + \frac{3}{4} q^2 \right) \right] = 4M^2 \left[ W_2 \left( \nu^2 - (\tau + \omega)^2 - \tau \right) + 2W_1 \tau \right] = 4EE' \cos^2 \frac{\theta}{2} \left[ W_2 + 2W_1 \tan^2 \frac{\theta}{2} \right].
\]

In going from the first to the second line we used $p \cdot k = M^2 \nu$, $k^2 = M^2 \tau$ and
\[
p_T^2 = p^2 - \frac{(p \cdot q)^2}{q^2} = M^2 \left( 1 + 2\omega + \tau + \frac{\omega^2}{\tau} \right) = \frac{M^2}{\tau} \left( \tau + \tau + \omega^2 \right),
\]
and to obtain the third line we exploited Eqs. (D.14) and (D.11). The resulting cross section, shown in Fig. 4.5, is
\[
\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}} \left[ W_2 + 2W_1 \tan^2 \frac{\theta}{2} \right].
\]

How does this compare to the limit of elastic scattering? From (4.21) and (4.29) we can write down the double-differential cross section for a pointlike fermion in the elastic case:
\[
\frac{d^2\sigma}{d\Omega dE'} = \frac{|M|^2}{4ME} \frac{1}{(4\pi)^2} \frac{E' \delta(\omega)}{2M^2} = \frac{\alpha^2}{4M^2 \tau^2} \frac{E' \delta(\omega)}{2M} (\nu^2 + \tau^2 - \tau) = \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}} \frac{\delta(\omega)}{2M} \left( 1 + 2\tau \tan^2 \frac{\theta}{2} \right).
\]

**Figure 4.5:** Double-differential inelastic $eN$ cross section from Eq. (4.57) at fixed lepton energy $E$ and scattering angle $\theta$. At large invariant masses, the resonance peaks are washed out. (Halzen and Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics, Wiley, 1984.)
Hence, the response functions reduce in the elastic limit to

\begin{align*}
W_1(\tau, \omega) &= \tau \frac{\delta(\omega)}{2M}, \\
W_2(\tau, \omega) &= \frac{\delta(\omega)}{2M}.
\end{align*}

(4.59)

The elastic peak in the cross section is clearly visible in Fig. 4.5. From comparing the two cross sections we also find that \(W_1\) encodes the spin of the target and vanishes for a spinless particle. For scattering on a composite nucleon, the results (4.59) in the elastic limit have to be multiplied with the Sachs form factor combinations in the Rosenbluth cross section (4.33). Finally, we can trade the dependence on \(\omega\) by a dependence on the Bjorken variable \(x\) if we use the relations

\begin{align*}
\tau &= \frac{\nu'}{2M} x, \\
\omega &= \frac{\nu'}{2M} (1 - x) \quad \Rightarrow \quad 2M W_1 = x \delta(1 - x) =: 2F_1, \\
\nu' W_2 &= \delta(1 - x) =: F_2.
\end{align*}

(4.60)

Here we defined the dimensionless structure functions\(^6\) \(F_1(\tau, x)\) and \(F_2(\tau, x)\). For elastic scattering on a pointlike particle they are only functions of \(x\); in addition, the \(\delta\)-function enforces \(x = 1\) in the elastic limit.

**Bjorken scaling and the parton model.** One might expect that for inelastic scattering processes \((x \neq 1)\), away from the nucleon resonance region, the structure functions \(F_1\) and \(F_2\) are complicated functions of \(\tau\) and \(x\). However, it turns out that in the deep inelastic domain \((\tau\) and \(\omega\) large) they are almost independent of \(\tau\) and only functions of \(\omega/\tau\), or equivalently \(x\):

\begin{align*}
F_{1,2}(\tau, x) &\approx F_{1,2}(x), \\
F_2(x) &= 2x F_1(x).
\end{align*}

(4.61)

This is called **Bjorken scaling** and visible in the left plot of Fig. 4.6. The second relation is the **Callan-Gross** relation which entails that \(F_1\) and \(F_2\) are not independent of each other. The origin of scaling is simply a dimensional argument that follows from the near scale invariance of massless perturbative QCD (up to logarithmic corrections). A dimensionless function can only depend on dimensionless variables; \(\tau\) and \(\omega\) are dimensionless because we scaled the momenta with the nucleon mass, which required the presence of a nonperturbative nucleon bound-state mass to begin with. If we scatter instead on (nearly) massless quarks, no such scale is available and therefore the dimensionless structure functions cannot depend on \(\tau\) and \(\omega\) individually but only on their dimensionless combination \(\tau/\omega \sim q^2/p \cdot q\). Hence, the observation of scaling is an indication for the composite nature of the nucleon in terms of pointlike, essentially massless quarks and gluons.

The experimental fact of Bjorken scaling has led to the development of the parton model. The proton is viewed as a collection of ‘partons’, namely valence quarks, sea quarks and gluons. The incoming momentum \(p_i\) of the proton (mass \(M\)) is the sum of the parton momenta: \(p_i = \sum_k p_k\), where \(p_k\) is the four-momentum of a single onshell parton with mass \(m_k\). The basic assumption we need in the following is collinearity:

---

\(^6\)Try not to confuse them with the Dirac and Pauli form factors from the last section. In fact, even their physical meanings are reversed: the structure function \(F_1\) carries the spin dependence, whereas in the form factor case it is rather the Pauli form factor (or the magnetic form factor \(G_M\)) that contains the nucleon spin.
\[ p_k = \xi_k p_i, \] which can be justified in the infinite momentum frame. If we generically write
\[ p_i = \left( \sqrt{p^2 + M^2} \right), \quad p_k = \left( \sqrt{\xi_k^2 p^2 + (p_k^\perp)^2 + m_k^2} \right), \]
then \( \xi_k \) defines the longitudinal momentum fraction of parton \( k \) in the direction of the proton’s three-momentum \( p \). In the infinite-momentum frame \((|p| \to \infty)\) we can neglect the transverse components and masses:\footnote{The relation \( m_k = \xi_k M \) is a bit nonsensical as it would imply that the ‘mass’ of a parton changes with its momentum fraction, but we need it for consistency of the naive parton model.}
\[ |p_k^\perp| \ll |p|, \quad m_k \approx \xi_k M \ll |p|, \quad \Rightarrow \quad p_k \approx \xi_k p_i, \quad \sum_k \xi_k = 1. \] (4.63)

The collinearity assumption allows for a simple interpretation of the Bjorken scaling variable. We have seen in Appendix D that elastic scattering on the nucleon corresponds to \( x = -q^2/(2p_i \cdot q) = 1 \). In the inelastic process \((x \neq 1)\), elastic scattering on a single parton \( k \) then entails that
\[ x_k := -\frac{q^2}{2p_k \cdot q} = \frac{x}{\xi_k} = 1 \quad \Rightarrow \quad \xi_k = x, \]
so that the Bjorken variable \( x \) assumes the meaning of the parton’s longitudinal momentum fraction in the infinite-momentum frame. The photon only couples to those partons whose momentum fraction is \( \xi_k = x \), hence a measurement of the structure function \( F_2(x) \) allows us to ‘see’ how the parton momenta are distributed inside the proton. In elastic scattering we have \( x = 1 \) and the photon couples to the whole proton since it carries the full momentum.

Let us define the momentum distribution \( f_k(\xi) \) of a parton in the hadron (the parton distribution function or PDF), so that \( f_k(\xi) \, d\xi \) is the probability density that a parton carries a momentum fraction between \( \xi \) and \( \xi + d\xi \). Momentum conservation implies
\[ \sum_k \int_0^1 d\xi f_k(\xi) = 1. \] (4.65)

Now suppose we scatter on spin-\( \frac{1}{2} \) quarks. Using the relations (4.60) together with \( x_k = x/\xi_k \) and \( m_k = \xi_k M \), the structure functions \( F_i^k \) for the parton \( k \) are
\[ 2F_1^k = 2MW_1^k = M \frac{m_k}{m_k} 2m_kW_1^k = M \frac{m_k}{m_k} x_k \delta(1 - x_k) = \delta(\xi_k - x), \]
\[ F_2^k = \nu'W_2^k = \delta(1 - x_k) = \frac{\xi_k^2}{x} \delta(\xi_k - x) = x \delta(\xi_k - x), \] (4.66)
and integrating over all partons yields
\[ F_1(x) = \sum_k e_k^2 \int d\xi f_k(\xi) F_1^k(\xi, x) \quad \Rightarrow \quad F_1(x) = \frac{1}{2} \sum_k e_k^2 f_k(x), \]
\[ F_2(x) = x \sum_k e_k^2 f_k(x). \] (4.67)
4.2 Deep inelastic scattering

Hence we have shown that in the parton model $F_1$ and $F_2$ are indeed only functions of $x$, and we can confirm the Callan-Gross relation (4.61). The latter is an experimental indication for the spin-1/2 nature of the quarks: if quarks had spin zero, $F_1(x)$ would vanish.

Due to the Callan-Gross relation only the structure function $F_2(x)$ will be relevant in the following. What will it look like? If the proton consisted of a single 'quark' that carried all of its momentum, the structure function would have a single peak at $x = 1$. If it consisted of three non-interacting quarks, the quarks would all carry the same momentum fraction and $F_2(x)$ would have a peak at $x = 1/3$. If the three quarks interact with each other, they can exchange momentum and hence the momentum fraction carried by each quark will fluctuate; the resulting structure function is a smooth distribution peaked near $x = 1/3$. Finally, the presence of sea quarks will lead to an enhancement at small $x$ because sea quarks are created in Bremstrahlung-like processes which are typically enhanced at small momenta and lead to $x f(x) \xrightarrow{x \to 0} \text{const}$. Note that gluons will also contribute to the momentum sum rule (4.65) whereas the structure function only probes electrically charged partons (quarks).
**Parton distribution functions.** Now let’s see how much information on the PDFs we can gather from experimental data on $F_2(x)$. There is no sensible way to distinguish two identical partons within a proton, but we can still group them according to the various quark and antiquark flavors: $f_k(x) = u(x), \bar{u}(x), d(x), \bar{d}(x), \ldots$, so that we have

$$
\frac{F_2^p(x)}{x} = q_u^2 (u(x) + \bar{u}(x)) + q_d^2 (d(x) + \bar{d}(x)) + q_s^2 (s(x) + \bar{s}(x)) + \ldots \quad (4.68)
$$

It is usually sufficient to stop at the strange quark because the probability for finding charm in the proton is negligible. $u(x)$ is the probability distribution for up quarks in the proton, $\bar{u}(x)$ that of anti-up quarks, and so on. One can also measure the structure function $F_2^n$ of the neutron via electron-deuteron scattering. Charge symmetry entails that the $d$ distribution in the neutron is identical to the $u$ distribution in the proton: $u = u^n = d^n$, $d = d^n = u^n$, $s = s^n = s^n$, and analogously for the antiquark PDFs:

$$
\frac{F_2^n(x)}{x} = q_d^2 ((u(x) + \bar{u}(x)) + q_d^2 (d(x) + \bar{d}(x)) + q_s^2 (s(x) + \bar{s}(x)) + \ldots \quad (4.69)
$$

In the following it will be more convenient to work with valence- and sea-quark distributions, defined via

$$
\begin{align*}
  u &= u_v + u_s, & \bar{u} &= \bar{u}_s, & s &= s_s, \\
  d &= d_v + d_s, & \bar{d} &= \bar{d}_s, & \bar{s} &= \bar{s}_s,
\end{align*}
$$

because antiquarks and strange quarks can only appear in the sea. Now, since the PDFs are number densities defined on the momentum fraction $x$, the integrals over this range are just the total flavor numbers of each quark type:

$$
\int_0^1 dx \, u_v(x) = 2, \quad \int_0^1 dx \, d_v(x) = 1, \quad \int_0^1 dx \left[ f_s(x) - \bar{f}_s(x) \right] = 0. \quad (4.71)
$$

The third relation expresses fermion number conservation for each flavor $f = u, d, s$; by summing over all individual partons, we must recover charge 1, baryon number 1 and strangeness 0 of the proton.

Can we extract the valence and sea distributions from the data for $F_2^{p,n}(x)$? We have two measured quantities but too many unknowns. Let’s make the further simplifying assumption that all sea-quark distributions are identical: $f_s(x) = \bar{f}_s(x) =: S(x)$. Then the structure functions for the proton and neutron become

$$
\begin{align*}
  \frac{F_2^p(x)}{x} &= q_u^2 \, u_v(x) + q_d^2 \, d_v(x) + (q_u^2 + q_d^2 + q_s^2) \, 2S(x), \\
  \frac{F_2^n(x)}{x} &= q_d^2 \, u_v(x) + q_u^2 \, d_v(x) + (q_u^2 + q_d^2 + q_s^2) \, 2S(x),
\end{align*}
$$

from where we can form their ratio and their difference:

$$
R = \frac{F_2^n}{F_2^p} = \frac{u_v + 4d_v + 12S}{4u_v + d_v + 12S}, \quad F_2^p - F_2^n = \frac{x}{3} \, (u_v - d_v). \quad (4.73)
$$
The ratio satisfies the Nachtmann inequality $\frac{1}{4} \leq R(x) \leq 4$: in a region of $x$ where the up (down) quarks dominate, we have $R = \frac{1}{4}$ ($R = 4$): if the sea quarks dominate we will find $R = 1$. The ratio is plotted in Fig. 4.6 and reveals that the sea quarks are indeed dominant at small $x$ whereas valence up quarks are important at large $x$. The difference in (4.73) is also shown in the figure: it measures only the valence-quark contribution and shows a peak around $x = 1/3$, as we had expected. Finally, the sum

$$\frac{9}{5} (F_2^p + F_2^n) = x \left( u_v + d_v + \frac{24}{5} S \right)$$

(4.74)

can be plugged into the momentum sum rule (4.65) which now takes the form

$$\int dx \, x \left( u_v + d_v + 6S \right) + \varepsilon = 1,$$

(4.75)

where $\varepsilon$ is the gluon contribution to the proton’s longitudinal momentum. From the experimental data we can roughly estimate

$$\frac{9}{5} \int dx \left( F_2^p + F_2^n \right) \approx 0.54 \approx 1 - \varepsilon,$$

(4.76)

which entails that the gluons carry almost half of the proton’s momentum. In fact, the gluon PDFs dominate at small values of $x$, see Fig. 4.7.

How good is the assumption that all sea-quark distributions are identical? If we go back to the original equations (4.68) and (4.69), take their difference and integrate over $x$, we have

$$\int dx \, \left( \frac{F_2^p - F_2^n}{x} \right) = \frac{1}{3} \int dx \left( u_v - d_v + u_s + \bar{u}_s - d_s - \bar{d}_s \right) \approx \frac{1}{3} + \frac{2}{3} \int dx \left( \bar{u}_s - \bar{d}_s \right)$$

(4.71)

Figure 4.7: Valence, sea-quark and gluon PDFs shown at two different resolution scales. Source: PDG, see Fig. 4.6.
which should equal $\frac{1}{3}$ if $\bar{u}_s = \bar{d}_s = S$ (this is the Gottfried sum rule). Instead, the experimental value is $\sim 0.23 \Rightarrow \int dx (\bar{d}_s - \bar{u}_s) \sim 0.15$, which entails that the light quark sea is indeed flavor-asymmetric.

**Scaling violations.** The left plot in Fig. 4.6 demonstrates that scaling is not exact because the structure functions exhibit a $Q^2$ dependence, which is most pronounced at small and large values of $x$. In terms of the PDFs, this implies that their $x$—dependence is not completely independent of the resolution scale $Q^2$ but also evolves with $Q^2$, which can be seen in Fig. 4.7. We can intuitively understand this as follows: a photon with intermediate $Q^2$ does not resolve the full spatial structure of the proton and mainly sees three interacting quarks, together with parts of the sea. In contrast, a high-$Q^2$ photon can resolve small distances and will reveal more and more of the quark sea which contains short-distance processes such as gluon emission from a quark or gluon splitting into $q\bar{q}$ pairs. As a result, the sea-quark contributions will be more prominent at higher $Q^2$. On the other hand, since the photon can resolve more partons, momentum conservation implies that each parton carries now a smaller fraction of the total momentum, and hence the PDFs will be shifted to smaller $x$. The resulting structure function $F_2(x)$ that sums up the individual quark PDFs will rise with higher $Q^2$ at small $x$ and fall with higher $Q^2$ at large $x$.

The short-distance dynamics depend on the resolution scale through the coupling $\alpha_s(Q^2)$. As a consequence, the individual quark structure functions $F_i^k$ will no longer be mere $\delta$—functions as in Eq. (4.66) but also inherit a $Q^2$ dependence from the coupling. Since the coupling is dimensionless, it also introduces a scale $\mu$ (the factorization scale), so that Eq. (4.67) becomes

$$F_i(x, Q^2) = \sum_k e_k^2 \int d\xi f_k(\xi, \mu) F_i^k(\xi, x, \frac{Q^2}{\mu^2}). \quad (4.77)$$

The $F_i^k$ encode the short-distance splitting processes and are calculable in perturbative QCD. The PDFs $f_k$, which now also depend on $\mu$, are inherently nonperturbative and have to be fitted to experimental data or calculated with nonperturbative methods. Since the nucleon structure function must be independent of the factorization scale $\mu$, its total derivative with respect to $\mu$ must vanish. Similarly to the Callan-Symanzik equation (1.87), one then derives the DGLAP equations (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi) $dF_i/d\mu = 0$ which relate PDFs at different $\mu$ with each other and thereby allow one to calculate the scaling violations using QCD perturbation theory.

**Compton amplitude.** How can PDFs be calculated nonperturbatively? The key idea is to relate the hadronic tensor $W^{\mu\nu}$ that enters the inelastic $eN$ cross section to the nucleon’s forward Compton scattering amplitude. The latter is measured in the process $\gamma N \rightarrow \gamma N$ and given by

$$T^{\mu\nu}(p, q) := i \int d^4 z e^{iqz} \langle N(p_i)| \mathcal{T} V_{\text{em}}^{\mu}(\frac{z}{2}) V_{\text{em}}^{\nu}(-\frac{z}{2}) |N(p_i)\rangle,$$  \quad (4.78)

where $V_{\text{em}}^{\mu}$ is the electromagnetic current operator from Eq. (2.183). The hadronic
4.2 Deep inelastic scattering

The tensor in Eq. (4.53) has the form

\[ 4\pi M W^\mu\nu(p, q) = \sum_X \frac{d^3p_f}{(2\pi)^3 2E_X} \langle N(p_i) | V^\mu_{em}(0) | X(p_f) \rangle \langle X(p_f) | V^\nu_{em}(0) | N(p_i) \rangle \times (2\pi)^4 \delta^4(q + p_i - p_f) \]

(4.79)

and encodes the electromagnetic transition from the nucleon to all possible final states. If we write the \( \delta \)-function in momentum space,

\[ (2\pi)^4 \delta^4(q + p_i - p_f) = \int d^4z \, e^{i(q+p_i-p_f)z}, \]

(4.80)

use Eq. (2.75) to shuffle the \( z \)-dependence of the phase factor \( e^{i(p_i-p_f)z} \) into the current operators, and sum over the complete set of states \( X \), we obtain:

\[
4\pi M W^\mu\nu(p, q) = \int d^4z \, e^{iqz} \langle N(p_i) | V^\mu_{em}(\frac{z}{\tau}) V^\nu_{em}(\frac{-z}{\tau}) | N(p_i) \rangle \\
= \int d^4z \, e^{iqz} \langle N(p_i) | [V^\mu_{em}(\frac{z}{\tau}), V^\nu_{em}(\frac{-z}{\tau})] | N(p_i) \rangle.
\]

(4.81)

In the second line we have replaced the product of the currents by their commutator because the matrix element of \( \sim V^\nu_{em}(\frac{-z}{\tau})V^\mu_{em}(\frac{z}{\tau}) \) is zero: it gives rise to a \( \delta \)-function \( \delta(q-p_i+p_f) \) which cannot be saturated by any intermediate state. Energy conservation would require \( E_X = M - E + E' = M - \nu' \leq M \), but the nucleon is the lightest ground-state baryon.

By means of the optical theorem, Eq. (4.81) can be written as the imaginary part of the forward Compton scattering amplitude: \( 4\pi M W^\mu\nu(p, q) = 2 \text{Im} T^{\mu\nu}(p, q) \). In the forward limit, the Lorentz-invariant dressing functions of the Compton amplitude depend on the same variables as the inelastic cross section, \( \tau \) and \( x \). However, whereas \( \tau \geq 0 \) for real or virtual photons, the variable \( x \) is no longer constrained to the interval \( x \in [0, 1] \) but can be arbitrary. The Compton amplitude has non-analyticities arising from intermediate baryon resonances and baryon-meson continua in the \( s \) and \( u \) channels, extending from

\[ M^2 \leq s < \infty, \quad M^2 \leq u < \infty \quad \Leftrightarrow \quad -1 \leq x \leq 1, \]

(4.83)

and the hadronic tensor is proportional to the imaginary part along the cut \( x \in [0, 1] \) in the complex \( x \) plane. Hence, a theoretical handle on the nucleon Compton scattering amplitude allows us also to compute the nucleon’s structure functions in DIS.

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*If we apply the kinematics of App. D to the Compton scattering process directly, then in the forward limit (vanishing momentum transfer \( q = 0 \)) we have \( p = p_i = p_f \) and the photon momentum is \( k = k_i = k_f \), so that \( k^2 \) and the crossing variable \( \nu \) are the independent Lorentz-invariants. The variables \( \tau \) and \( x \), as originally defined in DIS, are then given by

\[ \tau = -\frac{k^2}{4M^2}, \quad x = -\frac{k^2}{2p \cdot k} = \frac{2\pi}{\nu} \quad \Rightarrow \quad \{s, u\} = M^2 + 4M^2\tau \left( -1 \pm \frac{1}{x} \right) \]

(4.82)

and therefore \( s, u \geq M^2 \) implies \( |x| \leq 1 \).
Figure 4.8: Hadronic tensor $W^{\mu\nu}$ in the parton model, and its relation with the forward Compton scattering amplitude and its factorized handbag structure.

**Light cone expansion.** What about the PDFs? To begin with, it is important to realize that in the Bjorken limit the Fourier transform in Eq. (4.81) is dominated by the behavior close to the light cone $z^2 \to 0$, i.e., where the two interaction points are separated by a lightlike distance. This is easiest seen using light-cone variables:

$$a_\pm := \frac{1}{\sqrt{2}}(a_0 \pm a_3), \quad a_\perp = (a_1, a_2) \Rightarrow a \cdot b = a_+ b_+ + a_- b_- - a_\perp \cdot b_\perp. \quad (4.84)$$

Then the integral (4.81) becomes schematically:

$$W(p, q) = \int dz^- e^{i q_+ z^-} \int dz_+ e^{i q_- z_+} \int d^2 z_\perp e^{-i q_\perp \cdot z_\perp} W(p, z). \quad (4.85)$$

The domain of the $z_\perp$ integration is restricted since the current commutator vanishes outside the light cone ($z^2 = 2z_+ z_- - z_\perp^2 < 0$) due to causality. In light-cone variables, the Bjorken limit $\nu' \to \infty, x = \text{const.}$ corresponds to $q_+ \to \infty$ and $q_- = \text{const}$:

$$\sqrt{2} q_\pm = q_0 \pm q_3 \quad (4.9) \quad \nu' \pm \sqrt{\nu'^2 - q_0^2} = \nu' \left(1 \pm \sqrt{1 + \frac{2Mx}{\nu'}}\right) \approx \begin{cases} 2\nu' + Mx + \ldots \\ -Mx + \ldots \end{cases}$$

For $q_+ \to \infty$ and $q_- = \text{const}$, the integral (4.85) is determined by the behavior of the integrand for $z_- \to 0$ and $z_+ \text{ finite}$; this is the area with the least oscillations according to the Riemann-Lebesgue lemma. The condition $z_\perp^2 < 2z_+ z_- \text{ then implies } z^2 \to 0^+$ but $z_\mu \neq 0$, which is the light cone.

In order to proceed one has to work out the current commutator in Eq. (4.81). We derived equal-time current commutators earlier in Eq. (2.42) using the anticommutation relations (1.46) for the quark fields. For free fields one can generalize that formula to unequal times $x_0 \neq y_0$ with the generalized anticommutation relations

$$\{\psi(x), \bar{\psi}(y)\} = S(x - y), \quad \{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0, \quad (4.86)$$

where $S(z) := (i\partial + m) \Delta(z)$, and $\Delta(z)$ is the causal propagator\(^9\) that vanishes outside

\(^9\)In contrast to the Feynman propagator (2.82), the causal propagator sums up the positive- and negative-energy pole residues of a free scalar propagator.
the light cone, i.e., for spacelike distances \( z^2 < 0 \):

\[
\Delta(z) := \int \frac{d^3p}{2E_p} \frac{e^{-ipz} - e^{ipz}}{(2\pi)^3} |_{p^0 = E_p} = \int \frac{d^4p}{(2\pi)^3} e^{-ipz} \varepsilon(p^0) \delta(p^2 - m^2),
\]

and \( \varepsilon(a) = a/|a| = \Theta(a) - \Theta(-a) \) is the sign function. At equal times \( z_0 = 0 \), the causal propagators reduce to \( \Delta(z) = 0 \), \( \partial_0 \Delta(z) = -i\delta^3(z) \) and \( S(z) = \gamma_0 \delta^3(z) \) which reproduces the equal-time (anti-)commutation relations for scalar and fermion fields. Rederiving the current commutator relation in this case gives the result

\[
\left[ j^\Gamma_a(x), j^{\Gamma'}_b(y) \right] = i f_{abc} j^+_c(x, y) + d_{abc} j^-_c(x, y) + \frac{\delta_{ab}}{N} j^-(x, y),
\]

which depends on the bilocal currents

\[
j^\pm_a(x, y) := \frac{1}{2} \left( \overline{\psi}(x) \Gamma S(x - y) \Gamma' t_a \psi(y) \pm \overline{\psi}(y) \Gamma' S(y - x) \Gamma t_a \psi(x) \right).
\]

Here we can already recognize the handbag structure from Fig. 4.8 when putting the result back in the hadronic tensor \( W_{\mu\nu} \); for the electromagnetic current commutator we have \( \Gamma = \gamma^\mu \) and \( \Gamma' = \gamma^\nu \). The light-cone singularities come from the free propagator \( S(z) \) which for a massless fermion reduces to

\[
S(z) \xrightarrow{m=0} \frac{1}{2\pi} \phi \left( \varepsilon(z_0) \delta(z^2) \right).
\]

It represents the hard part of the process, namely the scattering of the photon on a single perturbative quark which was the underlying assumption of the parton model.

The soft part is expressed through the remaining matrix element of bilocal quark-antiquark currents which is closely related to the quantity in Eq. (4.2). One can work out the Dirac structures for \( \Gamma S(z) \Gamma' \) and \( \Gamma' S(-z) \Gamma \) and expand the resulting currents in Taylor series about \( z = 0 \). This leads to the \textit{operator product expansion} (OPE), schematically written as

\[
j \left( \frac{z^\pm_i}{z^2} \right) = \sum_i c_i(z) O_i(0),
\]

where the \( O_i(0) \) are local operators and the \( c_i(z) \) are the respective \textit{Wilson} coefficients. The operators which are most important at high \( Q^2 \) are those for which the \( c_i(z) \) are most singular as \( z^2 \rightarrow 0 \). This allows for a rigorous definition of PDFs that enter in Eq. (4.77) and makes them accessible for nonperturbative calculations.

Finally, the relation with the Compton amplitude also allows one to define non-forward \textit{generalized parton distributions} (GPDs). They encode the transverse structure of the proton which is related to the orbital momentum carried by the quarks and gluons. In contrast to the PDFs they are no longer connected with deep inelastic scattering because nonvanishing momentum transfer implies \( p_f \neq p_i \). Hence, they have to be extracted directly from deeply virtual Compton scattering (DVCS) or related processes.

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\textsuperscript{10}Extra care should be taken with regard to Schwinger terms, which include higher derivatives of the \( \delta \)-function and do not show up in commutators of zero components of currents.