Chapter 1

QCD

Quantum Chromodynamics (QCD) is the theory of the strong interaction. It describes the 'color force' that binds quarks and gluons to colorless hadrons (protons, neutrons, pions, etc.) and hadrons to nuclei. The strong force is ~ 100 times stronger than the electromagnetic interaction, extremely short-ranged (the typical interaction range is the size of a hadron $\sim 1\,\text{fm} = 10^{-15}\,\text{m}$), and its typical energy scale is the mass of the proton ~ 1 GeV.

The strong interaction is described by a local, non-Abelian $SU(3)_C$ gauge symmetry with several peculiar features. While quarks and gluons are asymptotically free at short distances, they are confined at large distances: only colorless bound states (hadrons) can be detected in experiments, and no quark or gluon has ever been observed directly. Nevertheless, nature has given us an abundance of evidence that these constituents exist, and their theoretical description in terms of a non-Abelian gauge theory has evolved from being considered a mere mathematical trick to a quite fundamental framework. In this chapter we will recapitulate the properties of QCD and its fundamental degrees of freedom and postpone the discussion of hadrons to Chapter 2.

1.1 QCD Lagrangian

Field content. The definition of a quantum field theory starts with constructing its Lagrangian $\mathcal{L}$ (or, equivalently, its action $S = \int d^4x\, \mathcal{L}$), based on the desired underlying symmetries. The symmetries of QCD are: Poincaré invariance, local color gauge invariance and various flavor symmetries, and the fields in the Lagrangian should transform under representations of these groups. The QCD Lagrangian contains quark and antiquark fields, and (as a consequence of color gauge invariance) gluon fields which mediate the strong interaction:

$$\psi_{\alpha,i,f}(x)\,, \qquad \bar{\psi}_{\alpha,i,f}(x)\,, \qquad A_a^\mu(x)\,. \tag{1.1}$$

The quark fields are Dirac spinors (index α) and transform under the fundamental representation of $SU(3)_C$ (color index $i = 1, 2, 3$ or red, green blue). The additional index $f = 1 \dots N_F$ labels the flavor quantum number (f = up, down, strange, charm, bottom, top). The eight gluon fields $A_a^\mu(x)$ are Lorentz vectors; there is one field for each generator t_a of the group ($a = 1 \dots 8$). In the fundamental representation:

$\mathsf{t}_a = \lambda_a/2$, where the λ_a are the eight Gell-Mann matrices; see Appendix A for a collection of basic $SU(N)$ relations. Gluons are flavor-blind and carry no flavor labels.

Gauge invariance. A free fermion Lagrangian $\overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi$ constructed from the quark and antiquark fields (we leave the summation over Dirac, color and flavor indices implicit) is invariant under global $SU(3)_C$ transformations

$$\psi'(x) = U\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)\,U^\dagger \qquad \text{with} \quad U = e^{i\varepsilon} = e^{i\sum_a \varepsilon_a \mathsf{t}_a}\,, \tag{1.2}$$

where the U_{ij} act on the color indices of the quarks. This invariance is no longer satisfied if we impose a local $SU(3)_C$ gauge symmetry $\psi'(x) = U(x)\psi(x)$ with spacetime-dependent group parameters $\varepsilon_a(x)$. The mass term is still invariant, but the derivative in the kinetic term acts now also on the spacetime argument of $U(x)$, and invariance of the Lagrangian (or the action) cannot be satisfied with an ordinary partial derivative. Hence, local color gauge invariance necessitates a covariant derivative and thus introduces gluon fields:

$$D_\mu = \partial_\mu - igA_\mu\,, \tag{1.3}$$

where $A^\mu(x) = \sum A^\mu_a(x)\,\mathsf{t}_a$ is an element of the Lie algebra. From the new Lagrangian $\overline{\psi}\,(iD\!\!\!\!/ - m)\,\psi$ we see that $D_\mu\psi$ must transform in the same way as the quark field itself. The required transformation property for the covariant derivative and the gluon field reads:

$$\overline{\psi}'\,D\!\!\!\!/'\psi' \stackrel{!}{=} \overline{\psi}\,D\!\!\!\!/\,\psi \quad \Rightarrow \quad D'_\mu\,\psi' = U D_\mu U^\dagger \psi' \tag{1.4}$$

$$\Rightarrow \quad A'_\mu = U A_\mu\, U^\dagger + \frac{i}{g}\,U(\partial_\mu U^\dagger). \tag{1.5}$$

The second term in A'_μ is particular to local gauge transformations; for a global symmetry we don't need a covariant derivative and could simply set $A_\mu = 0$. Note that we can generate gluon fields out of nothing ($A_\mu = 0$) by a local gauge transformation: such gauge fields $\sim U(\partial_\mu U^\dagger)$ are called pure gauge configurations.

Gluon dynamics. Next, we need a kinetic term that describes the dynamics of the gluons. We define the gluon field strength tensor as the commutator of two covariant derivatives:

$$F_{\mu\nu}(x) = \frac{i}{g}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]\,; \tag{1.6}$$

it is therefore also an element of the algebra. It inherits the transformation properties from (1.4): $F'_{\mu\nu} = U F_{\mu\nu} U^\dagger$. The contraction of two field-strength tensors is not gauge invariant; only its color trace is invariant due to the cyclic property of the trace:

$$\mathrm{Tr}\left\{F'_{\mu\nu}F'^{\mu\nu}\right\} = \mathrm{Tr}\left\{U F_{\mu\nu} U^\dagger\, U F^{\mu\nu} U^\dagger\right\} = \mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}. \tag{1.7}$$

Only the trace can therefore appear in the Lagrangian. We can write it as

$$\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} = F^a_{\mu\nu}\,F^{\mu\nu}_b\,\mathrm{Tr}\left\{\mathsf{t}_a\,\mathsf{t}_b\right\} = T(R)\,F^a_{\mu\nu}\,F^{\mu\nu}_a\,, \tag{1.8}$$

where $T(R) = 1/2$ in the fundamental representation of $SU(N)$, cf. Appendix A. From Eq. (1.5) we also conclude that a gluon mass term $\sim m_g\,A_\mu A^\mu$ cannot appear in the Lagrangian because it would violate gauge invariance: gluons must be massless.

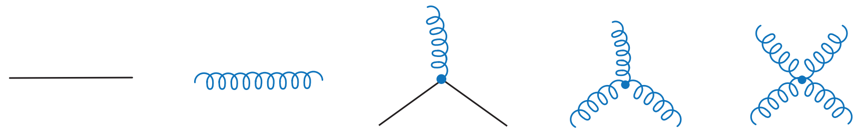

FIGURE 1.1: Tree-level propagators and interactions in the QCD action.

We can work out the components of the field-strength tensor as

$$\begin{aligned} F_{\mu\nu} = F^a_{\mu\nu}\,\mathsf{t}_a &= \partial_\mu A^a_\nu\,\mathsf{t}_a - \partial_\nu A^a_\mu\,\mathsf{t}_a - ig\,A^a_\mu A^b_\nu\,[\mathsf{t}_a,\mathsf{t}_b] \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}\,A^b_\mu\,A^c_\nu\right)\mathsf{t}_a\,, \end{aligned} \tag{1.9}$$

where we have used $[\mathsf{t}_a,\mathsf{t}_b] = i f_{abc}\,\mathsf{t}_c$. Note that in an Abelian gauge theory such as QED this commutator would vanish, leaving only the linear terms in the gluon fields. The non-Abelian nature of $SU(3)_C$ induces gluonic self-interactions which lead to significant complications. Inserting Eq. (1.9) into the term $F^a_{\mu\nu}\,F^{\mu\nu}_a$ and partial integration yields, up to surface terms in the action,

$$\begin{aligned} -\tfrac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a \;\cong\; & \frac{1}{2}\,A^a_\mu\,(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu)\,A^a_\nu \\ & - \frac{g}{2}\,f_{abc}\,(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)\,A^b_\mu\,A^c_\nu - \frac{g^2}{4}\,f_{abe}f_{cde}\,A^\mu_a\,A^\nu_b\,A^c_\mu\,A^d_\nu\,, \end{aligned} \tag{1.10}$$

from which the tree-level vertices can be read off. The first term $\sim A^2$ contains the inverse gluon propagator; we see already that it is proportional to a transverse projector in momentum space so we will have a problem with its inversion. The terms $\sim A^3$ constitute the three-gluon vertex and those proportional to A^4 the four-gluon vertex.

Let's consider the three-gluon vertex as an example: it must be Bose-symmetric; the structure constant f_{abc} is totally antisymmetric, and so the Lorentz remainder must be antisymmetric as well. To obtain the tree-level vertex in momentum space, write $f_{abc} = \frac{1}{3}\,(f_{abc} + f_{bca} + f_{cab})$ and rename color and Lorentz indices for the gluon fields so that $f_{abc}\,A^a_\mu\,A^b_\nu\,A^c_\rho$ can be pulled out. This yields for the second term in Eq. (1.10):

$$-\frac{ig}{6}\,f_{abc}\,A^a_\mu(k_1)\,A^b_\nu(k_2)\,A^c_\rho(k_3)\,\Big[(k_1-k_2)^\rho g^{\mu\nu} + (k_2-k_3)^\mu g^{\nu\rho} + (k_3-k_1)^\nu g^{\rho\mu}\Big], \tag{1.11}$$

with $k_1 + k_2 + k_3 = 0$. The four-gluon vertex is obtained analogously. By exploiting the Jacobi identity (A.3) for the structure constants, one arrives at the following momentum-space representation of the last term in Eq. (1.10), with $\sum_i k_i = 0$:

$$\begin{aligned} -\frac{g^2}{24}\,A^a_\mu(k_1)\,A^b_\nu(k_2)\,A^c_\rho(k_3)\,A^d_\sigma(k_4)\,\Big[\, & f_{abe}f_{cde}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}) \\ & + f_{ace}f_{bde}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\nu\rho}g^{\mu\sigma}) \\ & + f_{ade}f_{cbe}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma})\Big]\,. \end{aligned} \tag{1.12}$$

QCD action. The resulting QCD action has the most general form that is renormalizable, invariant under Poincaré transformations and local gauge transformations:

$$S_{QCD} = \int d^4x \, \mathcal{L}_{QCD} \,, \qquad \mathcal{L}_{QCD} = \bar{\psi}(x) \, (i\not{D} - \mathsf{M}) \, \psi(x) - \tfrac{1}{4} F^a_{\mu\nu} \, F^{\mu\nu}_a \,. \tag{1.13}$$

Again the summation over the Dirac, color and flavor indices of the quarks is implicit. M is the diagonal quark mass matrix to which we will return in a moment. Some further remarks:

- Eq. (1.13) also conserves charge conjugation and parity.
- In principle, another gauge-invariant and renormalizable (but parity-violating) term could appear in the Lagrangian, namely a topological charge density:

  $$\mathcal{Q}(x) = \frac{g^2}{8\pi^2} \, \mathrm{Tr} \left\{ F_{\mu\nu} \, \widetilde{F}^{\mu\nu} \right\} \qquad \text{with} \quad \widetilde{F}^{\mu\nu} = \frac{1}{2} \, \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \,, \tag{1.14}$$

  where $\widetilde{F}^{\mu\nu}$ is the dual field strength tensor. Since this can be written as the divergence of a current: $\mathcal{Q} = \partial_\mu K^\mu$, it contributes only a surface term to the action and in principle we could discard it (unless topological gauge field configurations play a role). Eq. (1.14) violates parity and would give rise to an electric dipole moment of the neutron, whose experimental upper limit is however tiny (which leads to the strong CP problem).
- We could have defined the gluon fields so that they absorb the coupling constant g (i.e., by replacing $A \to A/g$ and $F \to F/g$). From Eqs. (1.6), (1.10) and (1.13) we see that the only place in the Lagrangian where the coupling would then appear is in front of the gluon kinetic term, as a prefactor $1/g^2$. This shows that the sign of g is physically irrelevant.

Quark masses and flavor structure. With regard to the flavor structure, we can simply ignore the gluons since they are flavor independent. The quark-gluon interaction is flavor-blind, and the distinction between different quarks comes only from their masses. If the masses of all quark flavors were equal, the Lagrangian would exhibit an additional $SU(N_F)$ flavor symmetry. This is not realized in nature where we have

$$m_u \sim m_d \sim 2 \dots 6 \, \text{MeV}, \qquad m_s \sim 100 \text{ MeV}, \qquad \begin{array}{l} m_c \sim 1.3 \ \text{GeV}, \\ m_b \sim 4.2 \ \text{GeV}, \\ m_t \sim 173 \ \text{GeV}. \end{array} \tag{1.15}$$

The origin of these discrepancies is still unclear and comes from the electroweak sector, i.e., the Higgs mechanism. For our purposes, quark masses are an external input to QCD. They enter the QCD Lagrangian through the diagonal quark mass matrix in flavor space: $\mathsf{M} = \mathrm{diag}(m_1 \dots m_{N_F})$, which simply means that the flavor pieces in the Lagrangian add up, for example: $\bar{\psi} \, \mathsf{M} \, \psi = \sum_f m_f \, \bar{\psi}_f \, \psi_f$. The flavor structure of the Lagrangian is crucial for the properties of hadrons and we will return to it in Chapter 2.

Infinitesimal gauge transformations. For later convenience it is useful to work out the infinitesimal transformations of the fields. The covariant derivative as defined in Eq. (1.3) acts on fields that transform under the fundamental representations of $SU(3)_C$, i.e., the group elements. When acting on elements of the algebra (those containing the matrix generators t_a, for example ε, A_μ or $F_{\mu\nu}$), we need an additional commutator in its definition: $D_\mu = \partial_\mu - ig\,[A_\mu, \cdot\,]$, or written in components:

$$\begin{aligned}(D_\mu \varepsilon)^a &= (\partial_\mu \varepsilon - ig\,[A_\mu, \varepsilon])^a = \partial_\mu \varepsilon^a - ig\,A_\mu^c\,\varepsilon^b\,i f_{cba} \\ &= (\partial_\mu\,\delta_{ab} - g f_{abc}\,A_\mu^c)\,\varepsilon^b = D_\mu^{ab}\,\varepsilon^b\,.\end{aligned} \tag{1.16}$$

In the fundamental representation, the group generators are the Gell-Mann matrices; in the adjoint representation they are given by $\mathsf{t}^c_{ab} = -i f_{abc}$. Inserting this for Eq. (1.3), we see that the inner bracket in the last equation is indeed the covariant derivative D_μ^{ab} in the adjoint representation. In an Abelian gauge theory such as QED, the commutator vanishes and $D_\mu^{ab} = \partial_\mu\,\delta_{ab}$ is the ordinary partial derivative.

The infinitesimal gauge transformation of the fields is then given by

$$\begin{aligned}&\psi' = U\psi \approx (1 + i\varepsilon)\,\psi\,, \\ &\overline{\psi}' = \overline{\psi}\,U^\dagger \approx \overline{\psi}\,(1 - i\varepsilon)\,, \\ &A'_\mu = U A_\mu U^\dagger + \frac{i}{g}\,U(\partial_\mu U^\dagger) \approx A_\mu + i\,[\varepsilon, A_\mu] + \frac{1}{g}\,\partial_\mu\varepsilon = A_\mu + \frac{1}{g} D_\mu \varepsilon\,,\end{aligned} \tag{1.17}$$

from which we obtain:

$$\delta\psi = i\varepsilon\,\psi\,, \qquad \delta\overline{\psi} = -i\overline{\psi}\,\varepsilon\,, \qquad \delta A_\mu = \frac{1}{g} D_\mu\varepsilon\,, \qquad \delta F_{\mu\nu} = i\,[\varepsilon, F_{\mu\nu}]\,. \tag{1.18}$$

Classical equations of motion. The classical Euler-Lagrange equations of motion are obtained by taking functional derivatives of the classical QCD action $S[A, \psi, \overline{\psi}]$ with respect to the fields. This yields the classical Yang-Mills equation for the gluon fields,

$$\frac{\delta S}{\delta A_\nu^a} = \frac{\partial \mathcal{L}}{\partial A_\nu^a} - \partial_\mu\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^a)} = 0\,, \tag{1.19}$$

and similarly for the quark fields ψ and $\overline{\psi}$. While they are not directly relevant for our purposes, they will later enter in the quantum equations of motion and conservation laws. One arrives at the Dirac and Maxwell equations:

$$\begin{aligned}(i\partial\!\!\!/ + gA\!\!\!/ - \mathsf{M})\,\psi &= 0\,, \qquad & D_\nu\,F^{\mu\nu} &= g\,J^\mu\,, \\ \overline{\psi}\,(i\overleftarrow{\partial\!\!\!/} - gA\!\!\!/ + \mathsf{M}) &= 0\,, \qquad & D_\nu\,\widetilde{F}^{\mu\nu} &= 0\,.\end{aligned} \tag{1.20}$$

The equations for the gluon field-strength tensor are a direct generalization from electrodynamics, where the covariant derivative in the adjoint representation would reduce to the ordinary derivative. The current that appears on the right-hand side lives in the Lie algebra: $J^\mu = J^\mu_a\,\mathsf{t}_a$ with $J^\mu_a = \overline{\psi}\,\gamma^\mu\,\mathsf{t}_a\,\psi$, and it is covariantly conserved: $D_\mu J^\mu = 0$. The last identity in Eq. (1.20), $D_\nu\,\widetilde{F}^{\mu\nu} = 0$, is a consequence of the Bianchi identity $D_\nu F_{\alpha\beta} + D_\alpha F_{\beta\nu} + D_\beta F_{\nu\alpha} = 0$, which in turn follows from the Jacobi identity of the group generators, Eq. (A.3).