1.2 Quantization and Green functions

There are two standard methods to transform the classical action of Eq. (1.13) into a quantum field theory. One is the canonical formalism, where the fields are treated as operators on a Fock space and canonical (anti-)commutation relations are imposed. The other is the path-integral formalism where an integral over all fields is performed, thereby taking into account not only the classical field configurations but also their quantum corrections. Both methods are equivalent and we will use them in combination, depending on what better suits our needs. We will recall some basic concepts of quantum field theory with regard to simpler examples: a scalar field theory with only one species of fields, $\varphi(x)$, that is defined by the classical action $S[\varphi] = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$; and Quantum Electrodynamics (QED). Later we will generalize this to QCD.

Green functions. The central quantities of a quantum field theory are its n-point Green functions. They encode all possible interactions between n 'particles' described by the field φ, and they are related to the $S-$matrix elements of the theory. The Green functions of a scalar field theory are defined as

$$G(x_1, \dots x_n) := \langle 0 | \mathsf{T}\, \boldsymbol{\varphi}(x_1) \dots \boldsymbol{\varphi}(x_n) | 0 \rangle \,. \tag{1.21}$$

They are time-ordered vacuum expectation values of products of field operators $\boldsymbol{\varphi}(x_i)$ which are subject to spacelike commutation relations.

An alternative way to represent Green functions is the path-integral formalism; here they are given by

$$G(x_1, \dots x_n) = \frac{\int \mathcal{D}\varphi\, e^{iS[\varphi]}\, \varphi(x_1) \dots \varphi(x_n)}{\int \mathcal{D}\varphi\, e^{iS[\varphi]}} \,, \qquad \mathcal{D}\varphi = \prod_{i=1}^{n} d\varphi(x_i) \tag{1.22}$$

where the path integral measure goes over all possible field values at all space-time points. If we had different types of fields with additional group representation labels or Lorentz-Dirac indices, the product would go over all of them as well. In contrast to Eq. (1.21), the fields $\varphi(x_i)$ are here not operators but just ordinary functions of space-time (or anticommuting Grassmann fields when we are dealing with fermions). Eventually we will drop this distinction (operator or number) in the notation and assume that it becomes clear from the context.

When dealing with path integrals one should also remember to implement the correct imaginary-time boundary conditions on the space-time integrals: these are necessary to project out the interacting vacuum state that appears in the Green functions (1.21) (e.g. Peskin-Schroeder, p.284). Equivalently, one could add $i\varepsilon$ terms in the action which make the path integral well-defined and lead to the Feynman prescription for propagators. The most convenient solution is to formulate the theory in Euclidean space-time: in that case boundary conditions become irrelevant and the weight factor in the integrand is non-negative and defines a probability measure. This is usually done in practical calculations using the path-integral formalism. We will ignore this subtlety in what follows and assume that, in case of doubt, one can simply consult the Euclidean versions of all subsequent formulas.

1.2.1 Green functions in QED

QED action. The basic ideas about Green functions are best understood from QED, where they are directly related to experimentally measurable scattering processes. In contrast to QCD, it is an Abelian gauge theory and its Lagrangian is invariant under local $U(1)$ phase transformations:

$$\psi'(x) = U(x)\,\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)\,U^\dagger(x) \qquad \text{with} \quad U(x) = e^{i\varepsilon(x)}\,, \tag{1.23}$$

where $\varepsilon(x)$ is simply a number. The fields ψ and $\overline{\psi}$ represent electrons and positrons (one can add heavier lepton flavors such as the muon and the tau). The definition of the covariant derivative in Eq. (1.3) still holds, $D_\mu = \partial_\mu - igA_\mu$, except that also the photon field A^μ is just a spacetime-dependent number. Since the group elements commute, the transformed field is given by

$$A'_\mu = U A_\mu\, U^\dagger + \frac{i}{g}\, U(\partial_\mu U^\dagger) = A_\mu + \frac{1}{g}\,\partial_\mu\,\varepsilon\,. \tag{1.24}$$

The field-strength tensor reduces to

$$F_{\mu\nu}(x) = \frac{i}{g}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.25}$$

which is already gauge-invariant by itself. The resulting F^2 term does not contain any photon self-interactions but only the inverse photon propagator,

$$-\tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu} \cong \frac{1}{2}\,A_\mu\,(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu)\,A_\nu\,, \tag{1.26}$$

and the resulting action is given by

$$S_{QED} = \int d^4x\,\mathcal{L}_{QED}\,, \qquad \mathcal{L}_{QED} = \overline{\psi}(x)\,(i\not{D} - m)\,\psi(x) - \tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu}\,. \tag{1.27}$$

The only diagrams it encodes are the inverse tree-level fermion and photon propagators and the fermion-photon vertex:[1]

$$iS_0^{-1}(p) = \not{p} - m\,, \qquad i(D_0^{-1})^{\mu\nu}(p) = -p^2\,\delta^{\mu\nu} + p^\mu p^\nu\,, \qquad \Gamma_0^\mu(p,q) = ig\gamma^\mu\,. \tag{1.28}$$

The coupling constant $g = Qe$ is the electric charge ($Q = -1$ for electrons). The dimensionless coupling $\alpha_{\rm QED} = g^2/(4\pi)$ turns out to be small: if we insert dimensions,[2] we have

$$\alpha_{\rm QED} = \frac{e^2}{4\pi}\,\frac{1}{\hbar c\varepsilon_0} \approx \frac{1}{137}\,. \tag{1.29}$$

[1] We ignore several subtleties here for illustration purposes – we will discuss gauge fixing in Section 1.2.3 and renormalization in Section 1.3.

[2] In natural units $\hbar = 1$ sets the units of action $[ML^2/T]$, $c = 1$ sets the units of velocity $[L/T]$ and $\hbar c\varepsilon_0 = 1$ the units of charge $[C^2]$.

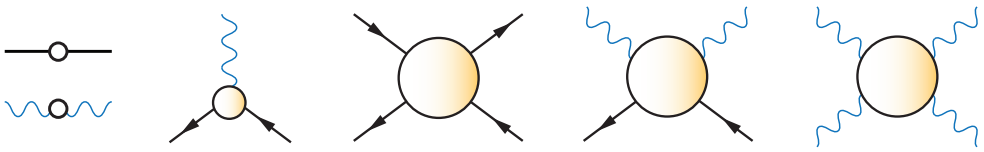


FIGURE 1.2: Lowest n-point functions in QED.

Green functions. Consider now the n-point Green functions in Fig. 1.2, which correspond to the definition in Eqs. (1.21–1.22): they are fully dressed and encode all quantum effects. We recover the dressed fermion propagator, the photon propagator ('vacuum polarization') and the dressed fermion-photon vertex, but also higher n-point functions without counterparts in the Lagrangian: the electron four-point function, its Compton scattering amplitude, the photon four-point ('light-by-light scattering') amplitude and so on. Their generic structure is determined from Lorentz covariance, for example for the simplest cases:

$$\begin{aligned} iS^{-1}(p) &= A(p^2)\,(\not{p} - M(p^2))\,, \\ i(D^{-1})^{\mu\nu}(p) &= (-p^2\,\delta^{\mu\nu} + p^\mu p^\nu)\,Z^{-1}(p^2)\,, \\ \Gamma^\mu(p,q) &= ig\,(f_1\gamma^\mu + f_2\,i\sigma^{\mu\nu}q_\nu + \dots)\,. \end{aligned} \tag{1.30}$$

The new scalar functions that appear here depend on the respective Lorentz invariants (p^2 for one independent momentum, p^2, q^2 and $p\cdot q$ for two momenta, etc.), and their number correlates with the number of independent tensor structures. $M(p^2)$ is the fermion 'mass function' and $Z(p^2)$ the photon dressing function. The fermion-photon vertex has 12 tensor structures; q is the photon momentum and p is the average momentum of the fermions, and $\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]$. Ignoring renormalization constants, these dressing functions recude at tree-level to

$$M(p^2) \to m, \qquad A(p^2) \to 1, \qquad Z(p^2) \to 1, \qquad f_1 \to 1, \qquad f_2 \to 0, \quad \dots \tag{1.31}$$

The same principles apply to the fermion four-point function: we can denote the three independent momenta by p, q and k, and there are 128 independent Dirac-Lorentz tensor structures:

$$\Gamma_{\alpha\beta\gamma\delta}(p,q,k) = g^2\,(h_1\,(\gamma^\mu)_{\alpha\beta}\,(\gamma_\mu)_{\gamma\delta} + \dots)\,. \tag{1.32}$$

Extracting observables. The point of all this is that, at least in the case of QED, the n-point functions are closely connected to measurable observables. The electron four-point function describes both the Møller ($e^-e^- \to e^-e^-$) and the Bhabha ($e^+e^- \to e^+e^-$) scattering processes; the electron-photon vertex encodes the electron's anomalous magnetic moment (which is related to f_2 above and would be zero without quantum corrections); the electron propagator contains the electron mass etc. The prescription how to relate n-point functions with observables is given by the LSZ formula (Lehmann, Symanzik, Zimmermann):

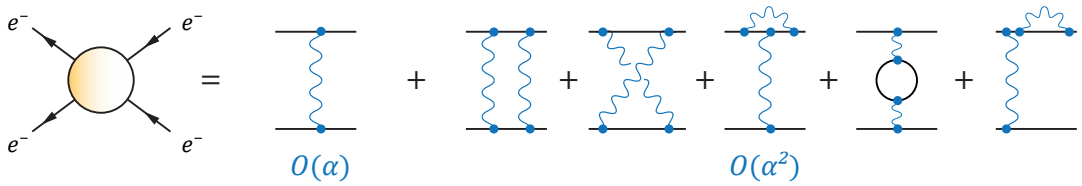


FIGURE 1.3: Lowest-order perturbative diagrams that contribute to Møller scattering. The last diagram does not survive the amputation.

- calculate the *connected* n-point function in quantum field theory (i.e., throw away the disconnected graphs where the particles do not interact);
- go to the kinematic limit where the external particles are onshell (e.g., $p_i^2 = m_e^2$ if p_i are external electron momenta);
- amputate external propagators and apply Dirac spinors to external fermion legs (and polarization vectors to photon legs) to get the *invariant amplitude* $\mathcal{M}$;
- take $|\mathcal{M}|^2$ to obtain the measurable cross section of the process.

In principle, the first step would be rather complicated but fortunately the smallness of $\alpha_{\rm QED}$ comes to rescue. If we perform a loop expansion for a given Green function, then higher loop diagrams come with higher powers of the coupling constant and in practice it is often sufficient to remain with the lowest (tree-level) order. For example in Møller scattering: the lowest-order contribution to the four-point function in Fig. 1.3 is the one-photon exchange diagram, which leads to the Mott cross section plus spin terms (more on that in Sec. 4.1). The smallness of $\alpha_{\rm QED}$ has caused the success of QED, where many observables can be calculated quite precisely simply by going to higher orders in perturbation theory. This leads to a variety of precision measurements of $\alpha_{\rm QED}$: from the anomalous magnetic moment ('$g-2$') of electrons and muons, from measurements of the Rydberg constant, the energy level splittings in atoms, etc.

When we try to apply the same principles to QCD we are confronted with two challenges that complicate matters enormously. On the one hand, the confinement property entails that it is pointless to calculate invariant scattering amplitudes of quarks and gluons because we can never measure such processes: what we can measure are reactions between hadrons (e.g. NN or $N\pi$ scattering), or hadrons that interact with leptons through the electroweak interaction (e^+e^- annihilation, eN scattering etc.). We will return to this point in Section 2.2, where we will discuss how one can still extract such measurable information from QCD's elementary Green functions. The second difficulty is that the coupling $\alpha_{\rm QCD}$ becomes large at low momenta (see Section 1.3) and invalidates a perturbative expansion. Unfortunately this is just the region that is relevant for hadron physics, so we must look for nonperturbative methods to calculate these n-point functions. This will be the task of the next subsection, and for illustration we will go back to the example of a scalar field theory.

1.2.2 Effective action and Dyson-Schwinger equations

Partition function. Exploiting the definition (1.22), a convenient method for generating Green functions is the partition function or generating functional

$$Z[J] = \int \mathcal{D}\varphi \, e^{i\left(S[\varphi] - \int_x \varphi(x) J(x)\right)} \,. \tag{1.33}$$

The idea is to add external source terms $J(x)$ to the action ($\int_x$ is a shorthand for $\int d^4x$), so that the Green functions are obtained as functional derivatives of $Z[J]$, where J is set to zero in the end:

$$G(x_1 \dots x_n) = \frac{i^n \delta^n}{\delta J(x_1) \dots \delta J(x_n)} \bigg|_{J=0} \frac{Z[J]}{Z[0]} \,. \tag{1.34}$$

Generalizing Eqs. (1.21), (1.22) and (1.34) to arbitrary polynomial functions of fields at different space-time points, one can replace their arguments by derivatives with respect to the sources:

$$\langle f(\varphi) \rangle := \langle 0 | \mathsf{T} \, f(\boldsymbol{\varphi}) | 0 \rangle = \frac{\int \mathcal{D}\varphi \, e^{iS[\varphi]} \, f(\varphi)}{\int \mathcal{D}\varphi \, e^{iS[\varphi]}} = f\left(\frac{i\delta}{\delta J}\right) \bigg|_{J=0} \frac{Z[J]}{Z[0]} \,. \tag{1.35}$$

In this way we can transform classical relations between fields, $f(\varphi) = 0$, into quantum identities: $\langle f(\varphi) \rangle = 0$.

In the following it will be convenient to leave the $J-$dependence intact, at least for intermediate steps in calculations. Green functions in the presence of the source J are then defined as

$$\langle f(\varphi) \rangle_J := \frac{\int \mathcal{D}\varphi \, e^{i\left(S[\varphi] - \int_x \varphi(x) J(x)\right)} \, f(\varphi)}{\int \mathcal{D}\varphi \, e^{i\left(S[\varphi] - \int_x \varphi(x) J(x)\right)}} = \frac{1}{Z[J]} \, f\left(\frac{i\delta}{\delta J}\right) Z[J] \,. \tag{1.36}$$

If we further set $Z[J] = e^{iW[J]}$ and exploit the relation $e^{-X} f(\partial) \, e^{X} = f(\partial + \partial X)$, we can express the last equation as

$$\langle f(\varphi) \rangle_J = f\left(-\frac{\delta W[J]}{\delta J} + \frac{i\delta}{\delta J}\right) , \tag{1.37}$$

which means that the dependence of the function f on each field value $\varphi(x_i)$ has to be replaced by a dependence on the bracket above, with $\delta/\delta J(x_i)$, and all unsaturated derivatives vanish since it acts on a constant.

1PI Green functions and effective action. The partition function $Z[J]$ generates all $n-$point Green functions. In practice, we are usually more interested in the connected Green functions and the one-particle-irreducible (1PI) Green functions. The former enter in S-matrix elements and are hence of physical interest, and the latter do away with the redundancy and describe the irreducible content of an $n-$point interaction vertex. (For example, renormalizability can be determined from the 1PI vertices alone.) One can also define generating functionals for these modified $n-$point functions: connected Green functions are derivatives of the functional $W[J] = -i \ln Z[J]$

with respect to J, and 1PI vertices are derivatives of the effective action $\Gamma[\widetilde{\varphi}]$, which is related to $W[J]$ via a Legendre transformation:

$$Z[J] = e^{iW[J]} = e^{i\left(\Gamma[\widetilde{\varphi}] - \int_x \widetilde{\varphi}(x)\, J(x)\right)} . \tag{1.38}$$

Instead of J, the effective action depends now on the averaged field $\widetilde{\varphi}$, which is the vacuum expectation value of φ in the presence of the source J:

$$\widetilde{\varphi}(x) \overset{(1.38)}{=} -\frac{\delta W[J]}{\delta J(x)} = \frac{i}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} \overset{(1.36)}{=} \langle \varphi(x) \rangle_J , \tag{1.39}$$

and vanishes in the 'physical' limit $J = 0$.[3] The meaning of 'effective action' and 'averaged field' becomes clear when comparing Eq. (1.38) with (1.33): the averaged field is the quantum expectation value of the classical field, and the effective action is the quantum averaged action, integrated over quantum fluctuations, with the path integral exponential as a weight factor. In the same way as the classical action $S[\varphi]$ contains the full content of the classical field theory, either of the functionals $Z[J]$, $W[J]$ or $\Gamma[\widetilde{\varphi}]$ determines the quantum field theory completely since all Green functions can be derived from them.

Let's use the following shorthand notation for functional derivatives of W and Γ:

$$W''_{xy}[J] = \frac{\delta^2 W[J]}{\delta J(x)\, \delta J(y)} , \qquad \Gamma''_{xy}[\widetilde{\varphi}] = \frac{\delta^2 \Gamma[\widetilde{\varphi}]}{\delta \widetilde{\varphi}(x)\, \delta \widetilde{\varphi}(y)} , \qquad \text{etc.} \tag{1.40}$$

Eq. (1.37) is useful to derive relations for connected Green functions. For example:

$$\begin{aligned} \langle \varphi(x)\, \varphi(y) \rangle_J &= \left(-W'_x[J] + \frac{i\delta}{\delta J(x)}\right)\left(-W'_y[J] + \frac{i\delta}{\delta J(y)}\right) \\ &= W'_x[J]\, W'_y[J] - i\, W''_{xy}[J] = \widetilde{\varphi}(x)\, \widetilde{\varphi}(y) - i\, W''_{xy}[J] . \end{aligned} \tag{1.41}$$

In the limit $J = 0$ the vacuum expectation value $\widetilde{\varphi}(x)$ vanishes, and we find that the 'connected' propagator $W''_{xy}[0]$ is just the usual propagator $\langle \varphi(x)\, \varphi(y) \rangle$ associated with the field φ.

Can we reformulate Eq. (1.37) so that it becomes an equation for the effective action and 1PI Green functions? From Eqs. (1.39) and (1.38) we see that the sources $J(x)$ and $\widetilde{\varphi}(x)$ are conjugated:[4]

$$W'_x[J] = -\widetilde{\varphi}(x) , \qquad \Gamma'_x[\widetilde{\varphi}] = J(x) , \tag{1.42}$$

and we find that the 1PI two-point function $\Gamma''_{xy}[0]$ is the inverse of $W''_{xy}[0]$:

$$\int_y W''_{xy}[J]\, \Gamma''_{yz}[\widetilde{\varphi}] \overset{(1.42)}{=} -\int_y \frac{\delta \widetilde{\varphi}(x)}{\delta J(y)}\frac{\delta J(y)}{\delta \widetilde{\varphi}(z)} = -\frac{\delta \widetilde{\varphi}(x)}{\delta \widetilde{\varphi}(z)} = -\delta^4(x - z) . \tag{1.43}$$

[3] Unless in the case of spontaneous symmetry breaking, but even then one can redefine the field so that its vacuum expectation value vanishes.

[4] Notice the similarity with thermodynamic systems: W and Γ are thermodynamic potentials; J is the 'intensive' and $\widetilde{\varphi}$ the 'extensive' variable. Differentiation with respect to one variable gives the conjugated one.

This explains why it is the *inverse* tree-level propagators, together with the tree-level vertices, that appear in the classical action: both correspond to the 1PI Green functions. Similarly, the three-point vertex is given by $\Gamma'''_{xyz}[0]$, the four-point vertex by $\Gamma''''_{xyzw}[0]$, and so on.

Since the dressed propagator in the presence of an external source plays a bit of a special role, we further abbreviate it by $\Gamma''_{xy}[\widetilde{\varphi}]^{-1} =: \Delta_{xy}[\widetilde{\varphi}]$. The product rule entails that the derivative of the propagator with respect to $\widetilde{\varphi}$ has the form:

$$\frac{\delta}{\delta\widetilde{\varphi}(z)}\,\Delta_{xy}[\widetilde{\varphi}] = -\int\!\!\!\int_{a\,b} \Delta_{xa}[\widetilde{\varphi}]\,\Gamma'''_{azb}[\widetilde{\varphi}]\,\Delta_{by}[\widetilde{\varphi}]. \tag{1.44}$$

Using Eq. (1.42) and

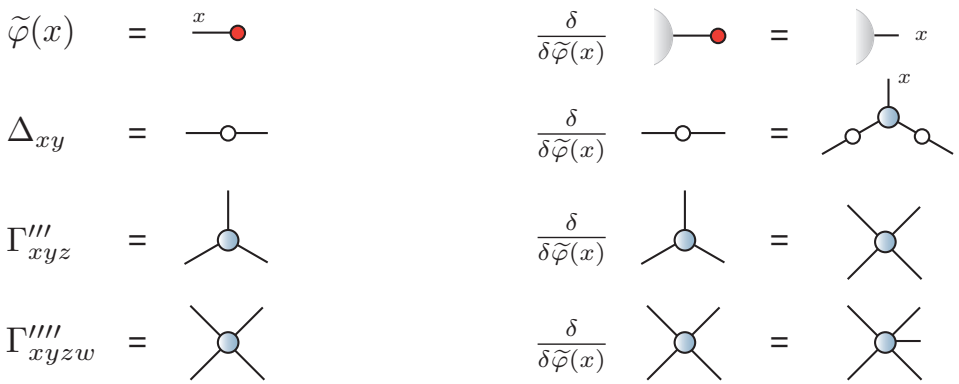
$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta\widetilde{\varphi}(y)}{\delta J(x)}\,\frac{\delta}{\delta\widetilde{\varphi}(y)} = -\int_y W''_{xy}[J]\,\frac{\delta}{\delta\widetilde{\varphi}(y)} = \int_y \Delta_{xy}[\widetilde{\varphi}]\,\frac{\delta}{\delta\widetilde{\varphi}(y)}\,, \tag{1.45}$$

we can now reexpress Eq. (1.37) in terms of the effective action and its derivatives:

$$\langle f(\varphi)\rangle_J = f\bigg(\widetilde{\varphi}(x) + \int_y \Delta_{xy}[\widetilde{\varphi}]\,\frac{i\delta}{\delta\widetilde{\varphi}(y)}\bigg). \tag{1.46}$$

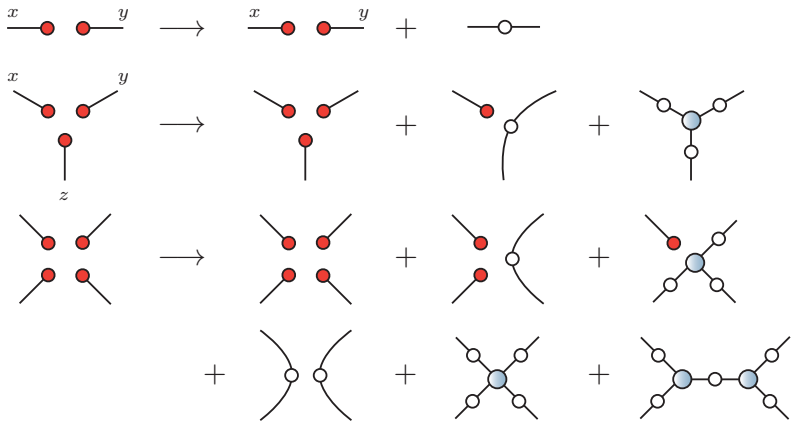
This is the identity that we were after, and it is extremely helpful in deriving relations for 1PI Green functions: if we want to evaluate a classical equation $f(\varphi) = 0$ for the fields at the quantum level, we only have to replace the (usually non-linear) dependence on φ by the bracket in (1.46) – which generates further derivatives and derivatives of propagators – and set all fields to zero in the end, together with all unsaturated derivatives. If the classical action contains more than one field, then the functional dependence in Eq. (1.46) holds for each $\widetilde{\varphi}_i(x)$, and the integral over y also goes over all intermediate (mixed!) propagators.

Graphical notation. In practice Eq. (1.46) amounts to repeated applications of the bracket with the derivative, which suggests to use a diagrammatic language. If we restrict ourselves to 1PI Green functions, we need graphical expressions for the source field $\widetilde{\varphi}(x)$, the propagator Δ_{xy}, the higher $n-$point functions Γ'''_{xyz}, Γ''''_{xyzw}, ... and the action of the functional derivative $\delta/\delta\widetilde{\varphi}(x)$ on these quantities:



and so on. In the graphical notation we will no longer distinguish between a Green function that depends on $\widetilde{\varphi}(x)$ as opposed to one where the field is set to zero, and we also suppress all prefactors and multiplicities that arise from derivatives.

What we need now is a graphical analogue for Eq. (1.46), where $\widetilde{\varphi}(x)$ is replaced by $\widetilde{\varphi}(x) + \int \Delta_{xy}\, i\delta/\delta\widetilde{\varphi}(y)$. If we work this out for products of fields we arrive at:



These graphs tell us how ordinary n-point functions are related with their 1PI counterparts: the second row is the three-point function $\langle\varphi(x)\varphi(y)\varphi(z)\rangle_J$ and the last expression is the four-point function. Upon setting $\widetilde{\varphi}(x) = 0$, the four-point function is the sum of a disconnected part, a 1PI term, and a 1-particle-reducible diagram that contains 1PI three-point functions. — We can interpret these diagrams also differently: without the extra integral term in Eq. (1.46) we would return to the classical quantity $f(\widetilde{\varphi})$. In the graphical notation we can then also drop the distinction between $\varphi(x)$ and its quantum expectation value $\widetilde{\varphi}(x)$ and use the same symbols for the fields that appear in the Lagrangian. Going from 'classical' to 'quantum' in the picture above then entails to connect the legs in all possible ways and equip them with dressed propagators.

Dyson-Schwinger equations. Dyson-Schwinger equations are the quantum equations of motion of a field theory. They follow from an invariance of the generating functional under a variation $\varphi(x) \to \varphi(x) + \epsilon(x)$ of the fields: since this is just a relabeling and all fields are integrated over, the path integral stays the same. Assuming that also the integral measure is invariant under this transformation, the condition $Z'[J] = Z[J]$ amounts to

$$\begin{aligned} Z'[J] &= \int \mathcal{D}\varphi'\, e^{i\left(S[\varphi'] - \int_x \varphi'(x) J(x)\right)} \\ &= \int \mathcal{D}\varphi\, e^{i\left(S[\varphi] - \int_x \varphi(x) J(x)\right) + i\int_x \epsilon(x)\left(\frac{\delta S[\varphi]}{\delta\varphi(x)} - J(x)\right)} \\ &= Z[J]\, \left\langle e^{i\int_x \epsilon(x)\left(\frac{\delta S[\varphi]}{\delta\varphi(x)} - J(x)\right)} \right\rangle_J \overset{!}{=} Z[J]\,, \end{aligned} \tag{1.47}$$

which yields the quantum average of the classical equations of motion in the presence of the source J:

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \right\rangle_J = J(x)\,. \tag{1.48}$$

This is the 'generating Dyson-Schwinger equation' because upon further differentiation it generates the tower of DSEs for QCD's Green functions. $\delta S/\delta\varphi$ is a function of the field $\varphi(x)$, so we can write the left-hand side in any of the forms (1.36), (1.37) or (1.46). The first version will result in an expression that contains (ordinary) Green functions for non-zero source terms, whereas the second form contains connected Green functions. If we apply further functional derivatives with respect to J and set $J = 0$ at the end, we successively obtain relations that couple the n-point functions of the theory among each other.

From Eqs. (1.42) and (1.46) we can read off the generating DSE for 1PI Green functions:

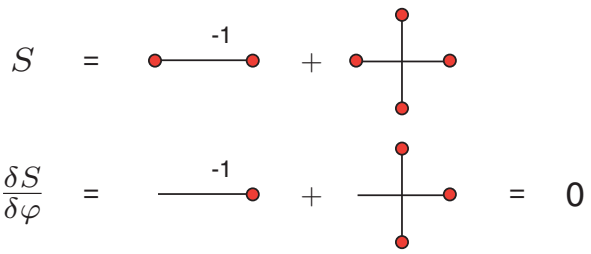
$$\Gamma'_x[\widetilde{\varphi}] = \frac{\delta S}{\delta\varphi}\left(\widetilde{\varphi}(x) + \int_y \Delta_{xy}[\widetilde{\varphi}]\,\frac{i\delta}{\delta\widetilde{\varphi}(y)}\right). \tag{1.49}$$

As before, the dependence on the field $\varphi(x)$ in the argument of $\delta S/\delta\varphi$ has to be replaced with the contents of the bracket, and all terms in the final expression which contain unsaturated derivatives vanish. The second term in the bracket generates the quantum corrections to the classical equations of motion: in its absence, the quantum effective action $\Gamma[\widetilde{\varphi}]$ would become identical to the classical action for the field $\widetilde{\varphi}$. Since the left-hand side is the first derivative of the effective action, further differentiation of Eq. (1.49) and finally setting $\widetilde{\varphi} = 0$ yields the system of DSEs for the 1PI n-point functions $\Gamma^{(n)}_{x_1,\ldots x_n}[0]$.

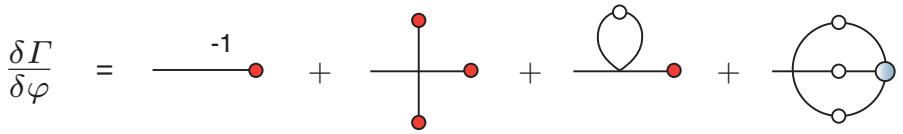
Dyson-Schwinger equations for ϕ^4 theory. Let's illustrate these relations by considering the simplest scalar field theory, ϕ^4 theory. The classical action and its functional derivative are given by

$$S = \int d^4x \left[\frac{1}{2}\left(\partial^\mu\varphi\,\partial_\mu\varphi - m^2\varphi^2\right) - \frac{g}{4!}\,\varphi^4\right], \qquad \frac{\delta S}{\delta\varphi} = -(\Box + m^2)\varphi - \frac{g}{3!}\,\varphi^3\,, \tag{1.50}$$

and setting $\delta S/\delta\varphi = 0$ yields the classical equations of motion. Diagrammatically, this amounts to



The line with '-1' is the inverse tree-level propagator. In the classical action it is contracted with the field φ; the functional derivative removes one instance of φ. As before we ignore all prefactors and multiplicities. Now 'connect the dots' in all possible ways to obtain the quantum equation of motion:

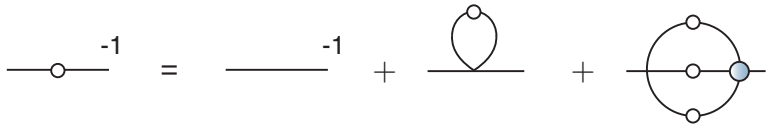


We arrive at the same result in formulas if we replace φ in Eq. (1.50) with the bracket in (1.49) and let it act on 1:

$$\Gamma'_x = -(\Box + m^2)\,\widetilde{\varphi}(x) - \frac{g}{3!}\left[\widetilde{\varphi}(x)^3 + 3i\,\widetilde{\varphi}(x)\,\Delta_{xx} + \iiint\limits_{z\,z'z''} \Delta_{xz}\Delta_{xz'}\Delta_{xz''}\Gamma'''_{zz'z''}\right], \quad (1.51)$$

where we have suppressed the arguments in $\Gamma'[\widetilde{\varphi}]$, $\Delta[\widetilde{\varphi}]$ and $\Gamma'''[\widetilde{\varphi}]$. Applying another derivative and setting $\widetilde{\varphi} = 0$ yields the DSE for the inverse scalar propagator:

$$\left.\frac{\delta^2\Gamma}{\delta\varphi^2}\right|_{\varphi=0} =$$



$$\Delta^{-1}_{xy} = -(\Box + m^2)\,\delta^4(x-y) - \frac{g}{3!}\left[3i\,\delta^4(x-y)\,\Delta_{xx} + \iiint\limits_{z\,z'z''} \Delta_{xz}\Delta_{xz'}\Delta_{xz''}\Gamma''''_{zz'z''y}\right]. \quad (1.52)$$

Note that $\Gamma'''_{xyz}[0] = 0$ because there is no three-point vertex in ϕ^4 theory. The equation states that the inverse dressed propagator is the sum of the inverse classical (tree-level) propagator plus quantum loop corrections. It is exact but depends on the four-point vertex which satisfies its own DSE.

Perturbation theory. The path-integral approach is of course also convenient for doing perturbative calculations. The idea is to split up the action into a non-interacting and an interacting part: $S[\varphi] = S_0[\varphi] + g\,S_I[\varphi]$. If the coupling constant g is small one can expand $e^{ig\,S_I[\varphi]}$ in powers of g. The standard procedure is to pull out the interacting part of the exponential from the path integral (1.33) by expressing it through a functional derivative, and to expand it in powers of the coupling constant:

$$Z[J] = e^{ig\,S_I\left[\frac{i\delta}{\delta J}\right]} \int \mathcal{D}\varphi\; e^{i\left(S_0[\varphi] - \int_x \varphi(x) J(x)\right)} = \sum_n \frac{(ig)^n}{n!}\left(S_I\left[\frac{i\delta}{\delta J}\right]\right)^n Z_0[J]\,, \quad (1.53)$$

where $Z_0[J]$ is calculable in a closed form. On the other hand, the resulting perturbation series for the propagator can also be obtained from Eq. (1.52): to get a DSE for the propagator instead of its inverse, multiply the equation with the tree-level propagator (Δ_0) from the left and the full propagator (Δ) from the right (or vice versa):

$$\Delta^{-1} = \Delta_0^{-1} - \Sigma \qquad \Rightarrow \qquad \Delta = \Delta_0 + \Delta_0\,\Sigma\,\Delta\,, \quad (1.54)$$

where Σ defines the self-energy. Reinsert the equation again for each instance of the dressed propagator on the r.h.s to generate a series. It is identical to the usual perturbation series from Eq. (1.53), however with the difference of being nonperturbative because there is always a remainder that reproduces the exact result, even if g is not small. Think for example of

$$f(x) = \frac{1}{1-x} = 1 + x f(x) = 1 + x + x^2 f(x) = \ldots, \quad (1.55)$$

which is valid for all x except $x = 1$, whereas the usual geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ converges to the exact result only for $|x| < 1$. Hence, the Dyson-Schwinger equation for a given Green function reproduces its perturbative expansion in the limit of a small coupling.

1.2.3 Green functions in QCD

Green functions in QCD. Let's now return to QCD. Instead of a single scalar field ϕ we have quark (ψ), antiquark ($\bar{\psi}$) and gluon fields (A^μ) which transform as Dirac spinors or vectors under Lorentz transformations and also carry color and flavor indices. The straightforward generalization of the partition function in Eq. (1.33) is:

$$Z[J,\eta,\bar{\eta}\,] = \int \mathcal{D}[A,\psi,\bar{\psi}\,]\; e^{i\left(S[A,\psi,\bar{\psi}\,]-\int_x (J_\mu A^\mu+\eta\,\bar{\psi}+\bar{\eta}\,\psi)\right)}\,. \tag{1.56}$$

The fermion fields in the path integral are now anticommuting Grassmann numbers, whereas their corresponding field operators in the canonical approach satisfy equal-time anticommutation relations:

$$\begin{aligned}\left\{\psi_{\alpha,i}(x),\psi^\dagger_{\beta,j}(y)\right\}_{x^0=y^0} &= \delta^3(\boldsymbol{x}-\boldsymbol{y})\,\delta_{\alpha\beta}\,\delta_{ij},\\ \left\{\psi_{\alpha,i}(x),\psi_{\beta,j}(y)\right\}_{x^0=y^0} &= \left\{\psi^\dagger_{\alpha,i}(x),\psi^\dagger_{\beta,j}(y)\right\}_{x^0=y^0} = 0\,.\end{aligned} \tag{1.57}$$

For example, the quark propagator is given by

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|\mathsf{T}\,\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)|0\rangle = \frac{i^2\delta^2}{\delta\bar{\eta}(x_1)\,\delta\eta(x_2)}\bigg|_{J,\,\eta,\,\bar{\eta}=0}\frac{Z[J,\eta,\bar{\eta}\,]}{Z[\,0,0,0\,]}\,. \tag{1.58}$$

It is gauge dependent since the propagator contains fields at different space-time points, and performing a gauge transformation does not leave it invariant.

Similarly, we would write the gluon propagator as

$$D^{\mu\nu}(x_1,x_2) = \langle 0|\mathsf{T}\,A^\mu(x_1)A^\nu(x_2)|0\rangle = \frac{i^2\delta^2}{\delta J_\mu(x_1)\,\delta J_\nu(x_2)}\bigg|_{J,\,\eta,\,\bar{\eta}=0}\frac{Z[J,\eta,\bar{\eta}\,]}{Z[\,0,0,0\,]}\,, \tag{1.59}$$

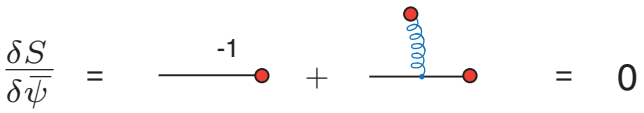
but unfortunately this expression does not yet make sense. We can read off the problem from the kinetic gluon term in the Lagrangian (the first term in (1.10)):

$$-\tfrac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a \;\cong\; \frac{1}{2}\,A^a_\mu\,(\Box\,g^{\mu\nu}-\partial^\mu\partial^\nu)\,A^a_\nu+\ldots \tag{1.60}$$

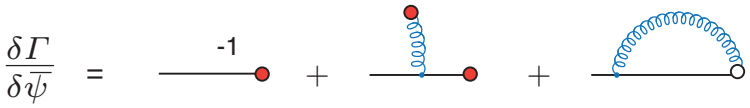
Since the inverse tree-level propagator in the bracket is a transverse projector, it has zero modes, namely all longitudinal field configurations of the form $A^a_\nu = \partial_\nu\Theta^a$. It cannot be inverted, and the propagator does not exist. Phrased differently, the path integral overcounts physically equivalent gauge degrees of freedom: if we split the gluon field into transverse and longitudinal components: $A^\mu = A^\mu_T + A^\mu_L$, with $A^\mu_L = \partial^\mu\Theta$, then only the transverse fields contribute to the propagator term in (1.60). In QED, the longitudinal parts would drop out completely from $-\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu}$ as it is determined by the inverse photon propagator alone – they represent the 'spurious' gauge degrees of freedom which still appear in the path integral $\int\mathcal{D}A$. The analogous problem in the canonical formalism is that the propagator cannot be canonically quantized because the time component of its conjugate momentum $F_{\mu0}$ vanishes.

Dyson-Schwinger equations. Before taking care of these issues, let's pretend for the moment that the gluon propagator is well-defined and let's see whether our earlier derivation of DSEs also works for QCD. The classical action of QCD takes the form:

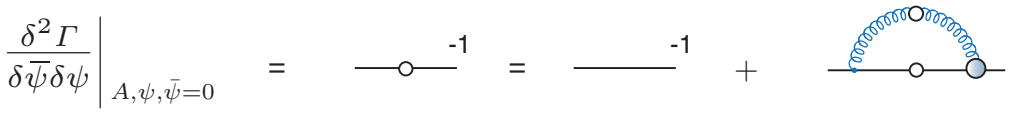
We use the convention that the left blobs in the fermion terms represent the antiquark fields $\bar{\psi}$ and the right blobs the quark fields ψ. Taking a functional derivative with respect to $\bar{\psi}$ yields the classical equation of motion for the quark, the Dirac equation:



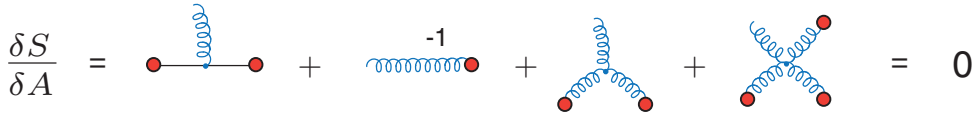
Consequently, the quantum equation of motion becomes (connect the dots)



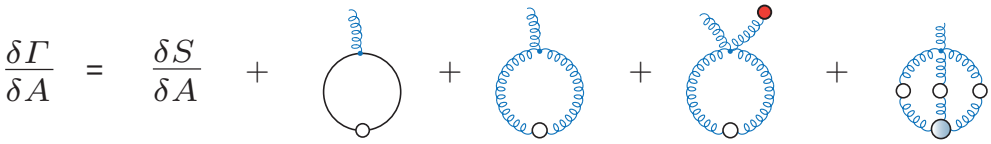
Note that the equation includes now a mixed, field-dependent quark-gluon propagator on the right. Taking another functional derivative with respect to ψ and setting all fields to zero gives us the DSE for the inverse quark propagator:



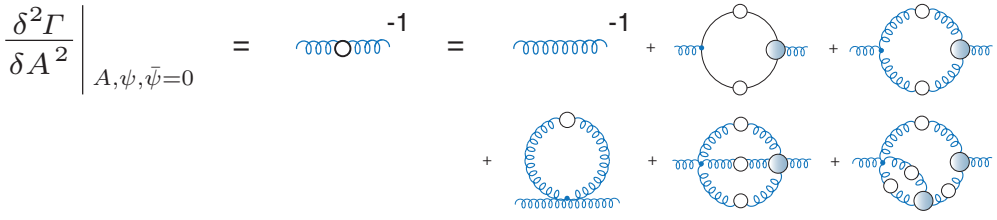
In principle we can repeat the same steps also for the gluon DSE. Derive the classical equation of motion for the gluon (the Maxwell equation),



and the corresponding quantum equation of motion:



Taking another derivative w.r.t. A and setting all sources to zero yields the gluon DSE:



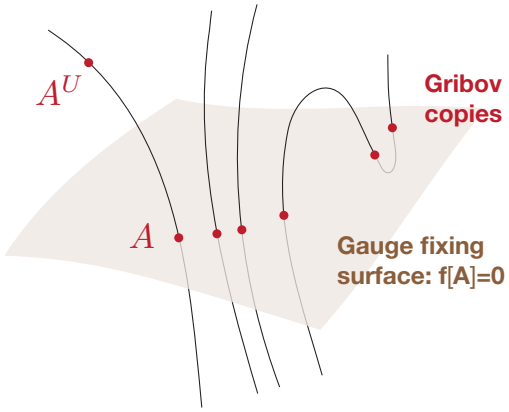


FIGURE 1.4: Gauge orbits and gauge-fixing surface.

It expresses the inverse dressed propagator in terms of the inverse classical propagator plus quantum-loop corrections. Actually we have ignored another subtlety: if the action contains several fields, one has to sum over them when taking the derivative of the propagators in Eq. (1.44), which also leads to mixed vertices. The general rule is that, after taking functional derivatives, for each internal 'half-propagator' that is connected to a dressed vertex one has to sum over all types of fields. This does not modify the quark and gluon DSEs but it will produce additional diagrams, for example, in the quark-gluon vertex DSE. — Irrespective of that, the gluon propagators in the equation are not yet well-defined, so we should have started from the gauge-fixed action including the ghost terms. This is simple to implement and merely leads to additional ghost diagrams, for example an additional ghost loop in the gluon DSE that resembles the quark loop. But how does it come about?

Faddeev-Popov gauge fixing. The standard method is the Faddeev-Popov gauge fixing procedure. Let's denote a gauge transformation of the gluon field by $A \to A^U$, where U is some gauge transformation with gauge parameter ε. Impose a gauge-fixing function $f[A]$ which we want to set to zero at the end: $f[A] = 0$, to single out a hypersurface of fixed gauge (cf. Fig. 1.4). Define the Faddeev-Popov operator $M[A]$ as the derivative of the gauge-fixing condition with respect to the gauge transformation parameter:

$$M[A] := \left.\frac{\delta f[A^U]}{\delta \varepsilon}\right|_{f[A^U]=0} . \tag{1.61}$$

$M[A]$ does not depend on the gauge transformation U. For example, a linear covariant gauge is defined by $f[A] = \partial_\mu A^\mu$, and from Eq. (1.18) we have $\delta A^\mu = \frac{1}{g} D^\mu \varepsilon$, so that the Faddeev-Popov operator in this case is given by[5]

$$M[A]_{ab}(x,y) = \frac{1}{g}\, \partial_\mu D^\mu_{ab}\, \delta^4(x-y) . \tag{1.62}$$

[5] In QED, this expression is also independent of A.

Consider now the 'functional unity'

$$\int \mathcal{D}U \, \det M[A] \, \delta(f[A^U]) = 1 \,, \tag{1.63}$$

which is the infinite-dimensional continuum version of the one-dimensional relation[6]

$$\int_{-\infty}^{\infty} d\varepsilon \left| \frac{df(\varepsilon)}{d\varepsilon} \right|_{f(\varepsilon)=0} \delta(f(\varepsilon)) = \int_{-\infty}^{\infty} d\varepsilon \, |f'(\varepsilon_0)| \, \frac{\delta(\varepsilon - \varepsilon_0)}{|f'(\varepsilon_0)|} = 1 \,. \tag{1.64}$$

The $\delta-$function in (1.63) is an infinite product of $\delta-$functions at each space-time point x, and $\mathcal{D}U$ is the group measure. We can insert Eq. (1.63) in the path integral:

$$Z = \int \mathcal{D}U \int \mathcal{D}A \, \det M[A] \, \delta(f[A^U]) \, e^{iS[A]} \,, \tag{1.65}$$

and, since Z is gauge-invariant, perform a gauge transformation $A^U \to A$. The gauge field measure $\mathcal{D}A$, the group measure $\mathcal{D}U$, the Faddeev-Popov determinant and the classical action $S[A]$ are all invariant under this operation, so that it merely amounts to replacing $\delta(f[A^U]) \to \delta(f[A])$. The integrand then depends no longer on U and the group integration $\mathcal{D}U$ factorizes; it produces an infinite constant which drops out whenever we normalize Z, for example when calculating Green functions. The remaining $\delta-$function restricts the integration over all fields to the hypersurface $f[A] = 0$. Each gauge orbit contributes only one field configuration and we have an integration over physically distinct fields.

The remaining goal is to shuffle the Faddeev-Popov determinant and the $\delta-$function in (1.65) into the action, at the price of introducing new, unphysical fields which are merely a consequence of fixing the gauge. We can take care of the $\delta-$function by changing the gauge fixing condition to $f[A] + \frac{\xi}{2} B = 0$, where $B(x)$ lives in the Lie algebra but does not depend on A. This does not affect the Faddeev-Popov determinant, but the functional integral Z_B depends now implicitly on B. Since any B leads to the same gauge-invariant physics, we can work with Z_B, $Z_{B'}$ or $\int \mathcal{D}B \, F(B) \, Z_B$; these are all equivalent. If we integrate over the functions $B(x)$ with some Gaussian weight, we can remove the $\delta-$function in favor of a new term in the action:

$$\begin{aligned} Z &= \int \mathcal{D}B \, e^{-\frac{i\xi}{8}\int d^4x \, B^2(x)} \, \mathcal{D}A \, \det M[A] \, \delta\big(f[A] + \tfrac{\xi}{2} B\big) \, e^{iS[A]} \\ &= \int \mathcal{D}A \, \det M[A] \, e^{\,i\left(S[A] - \int d^4x \, \frac{f[A]^2}{2\xi}\right)} . \end{aligned} \tag{1.66}$$

For example, with a linear covariant gauge this provides a welcome modification to the propagator term in Eq. (1.10), as it is no longer transverse in momentum space but

[6] Here we have assumed that the gauge-fixing condition is unique, i.e., that the equation $f(\varepsilon) = 0$ admits only one solution ε_0. This is usually not the case due to Gribov copies: the gauge-fixing condition can intersect the gauge orbits more than once and is therefore not complete. In this case both $f[A] = 0$ and $f[A^U] = 0$ are realized on the same gauge orbit, which means that the FP operator has zero eigenvalues. The problem does not appear in QED where the residual gauge freedom can be removed by imposing appropriate boundary conditions on the fields.

has instead the form

$$-\tfrac{1}{4}F^a_{\mu\nu}\,F^{\mu\nu}_a-\frac{(\partial_\mu A^\mu_a)^2}{2\xi}\;\cong\;\frac{1}{2}\,A^a_\mu\left(\Box\,g^{\mu\nu}-\partial^\mu\partial^\nu+\frac{1}{\xi}\,\partial^\mu\partial^\nu\right)A^a_\nu+\dots \tag{1.67}$$

and can be inverted. ξ is the gauge parameter: $\xi=0$ defines the Landau gauge, $\xi=1$ the Feynman gauge, and there are many other possible choices which differ not only by the gauge parameter but also by the gauge fixing condition (Coulomb gauge, axial gauge, light-cone gauge, maximal Abelian gauge etc.).

Finally we want to shift the Faddeev-Popov determinant into the action as well. We can write the determinant as a functional integral,

$$\det M[A]=\int\mathcal{D}c\,\mathcal{D}\bar{c}\;e^{-\iint\limits_{xy}\bar{c}_a(x)\,M[A]_{ab}(x,y)\,c_b(y)}\,, \tag{1.68}$$

where the Faddeev-Popov ghosts $c^a(x)$, $\bar{c}^a(x)$ are scalar but Grassmann-valued fields. They carry the wrong Bose-Fermi statistics, but this is of no concern since they are unphysical anyway. In conclusion, we have found that introducing a gauge-fixing term also introduces ghost fields.[7] This leads to new Green functions; from Eq. (1.68) we see that the Faddeev-Popov operator defines the inverse tree-level ghost propagator.

Reinserting the quarks and including all source terms, the final partition function for QCD assumes the form

$$Z[J,\eta,\bar{\eta},\sigma,\bar{\sigma}\,]=\int\mathcal{D}[A,\psi,\overline{\psi},c,\bar{c}\,]\;e^{i\left(S[A,\psi,\overline{\psi}\,]+S_{\rm GF}[A,c,\bar{c}\,]+S_{\rm C}\right)}, \tag{1.69}$$

where the gauge-fixing part of the action is

$$S_{\rm GF}=-\int\limits_x\frac{f[A]^2}{2\xi}+\iint\limits_{x\,y}i\bar{c}_a\,M[A]_{ab}\,c_b=\int\limits_x\left(-\frac{(\partial_\mu A^\mu_a)^2}{2\xi}+\frac{i}{g}\,\bar{c}_a\partial_\mu D^\mu_{ab}\,c_b\right). \tag{1.70}$$

The second equality holds for a linear covariant gauge; the factor i/g can be absorbed in the ghost fields. The source term reads

$$S_{\rm C}=-\int\limits_x\left(J_\mu A^\mu+\overline{\psi}\,\eta+\bar{\eta}\,\psi+\bar{\sigma}\,c+\bar{c}\,\sigma\right), \tag{1.71}$$

where η, $\bar{\eta}$ are the quark sources and σ, $\bar{\sigma}$ the ghost sources.

BRST symmetry. A more economical way to arrive at Eq. (1.70) is to impose BRST invariance of the action (Becchi, Rouet, Stora, Tyutin). Consider an infinitesimal gauge transformation (1.18) where the gauge parameter is a ghost field $c(x)=c_a(x)\,\mathtt{t}_a$, i.e., a scalar anticommuting Grassmann field:

$$\delta\psi=ic\,\psi\,,\qquad\delta\overline{\psi}=-i\overline{\psi}\,c\,,\qquad\delta A_\mu=\frac{1}{g}D_\mu c\,,\qquad\delta F_{\mu\nu}=i\,[c,F_{\mu\nu}]\,. \tag{1.72}$$

[7] In QED (at least with linear gauges), the ghosts are not dynamical because the Faddeev-Popov determinant is $A-$independent and can be pulled out of the path integral.

If we further demand that this transformation be nilpotent ($\delta^2 = 0$), it is straightforward to prove that any of the relations above fixes the transformation behavior of the ghost itself: $\delta c = \frac{i}{2}\,[c, c]$ or, in components: $\delta c_a = -\frac{1}{2}\,f_{abc}\,c_b\,c_c$. (Nilpotency of this last relation can be shown using the Jacobi identity; note that δ also anticommutes with c). Applying δ increases the ghost number (the charge corresponding to a $U(1)$ symmetry of the ghost fields) by one unit; hence, when applied to the antighost, it must produce a scalar field with ghost number zero, the so-called Nakanishi-Lautrup field: $\delta\bar{c} =: -iB$. Nilpotency of the antighost transformation then fixes $\delta B = 0$. The different treatment of c and $\bar{c}$ implies that they are not conjugates of each other but truly independent fields.

Since the classical action $S[A,\,\psi,\,\overline{\psi}\,]$ is gauge invariant and BRST is a gauge transformation, it is also BRST invariant. The most general BRST-invariant action is then the sum of the classical action plus a term $S_{\mathrm{GF}} = \delta\mathcal{O}$ which is a BRST variation itself, since in that case we have $\delta S_{\mathrm{GF}} = \delta^2\mathcal{O} = 0$. Adding this to the action means fixing a gauge; which gauge we get depends on $\mathcal{O}$. To recover (1.70), we contract the antighost with our earlier gauge-fixing condition $f[A] + \frac{\xi}{2}\,B$:

$$S_{\mathrm{GF}} = \delta \int\limits_x i\bar{c}_a \left(f_a[A] + \frac{\xi}{2}\,B_a \right) = \int\limits_x B_a \left(f_a[A] + \frac{\xi}{2}\,B_a \right) + \iint\limits_{x\;y} i\bar{c}_a\,M[A]_{ab}\,c_b. \quad (1.73)$$

Inserting the equations of motion for B_a, namely $f_a + \xi B_a = 0$, yields again Eq. (1.70); the same result follows from integrating over B_a in the path integral. Hence, imposing BRST invariance simultaneously generates gauge-fixing and ghost terms in the action.

Ward-Takahashi and Slavnov-Taylor identities. Green functions are not gauge invariant, but the gauge invariance of the generating functional (1.69) can be used to derive identities for them. As in the derivation of Dyson-Schwinger equations, a gauge transformation under the path integral is just a relabeling of fields, so Z is invariant; also the classical action is gauge invariant. The only gauge-dependent terms in (1.69) are then S_{GF} and S_C, and as in the derivation in Eq. (1.47), invariance of Z leads to the relation

$$\langle\, \delta S_{\mathrm{GF}} + \delta S_{\mathrm{C}}\,\rangle_J = 0\,, \quad (1.74)$$

which represents the generic form of a Ward-Takahashi identity. Again, with Eqs. (1.36), (1.37), (1.42) and (1.46) one can write this as a master equation for full Green functions (with sources J and derivatives $Z'[J]$), connected Green functions (J and $W'[J]$) or 1PI Green functions (with sources $\widetilde{\varphi}$ and derivatives of the effective action, $\Gamma'[\widetilde{\varphi}]$).

In the case of non-Abelian gauge theories it is more convenient to exploit BRST invariance. Here δS_{GF} vanishes as well and only the BRST variations of the fields in the source term remain to be evaluated: $\langle \delta S_{\mathrm{C}} \rangle_J = 0$. In the compact notation from earlier:

$$\langle \delta S_{\mathrm{C}} \rangle_J = \int\limits_x \sum_i J_i\,\langle \delta\varphi_i \rangle_J = \int\limits_x \sum_i \frac{\delta\Gamma}{\delta\widetilde{\varphi}_i}\,\langle \delta\varphi_i \rangle_J = 0\,, \quad (1.75)$$

where the $\delta\varphi_i$ are now nonlinear functions of the fields themselves, cf. Eq. (1.72). This leads to the Slavnov-Taylor identities.