1.3 Running coupling and renormalization

In our discussion so far we have bypassed the problem of renormalization entirely. The need for renormalization is related to the behavior of a theory at infinitely large energies or infinitesimally small distances. In practice it becomes visible in the perturbative expansion of Green functions. Take for example the tadpole diagram in φ^4 theory,

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2}\,, \tag{1.76}$$

which diverges for $k \to \infty$. As we will see below, renormalizability means that the coupling constant of the theory (or the coupling constants, if there are several of them) has zero or positive mass dimension: $d_g \geq 0$. This can be intuitively understood as follows: if M is the mass scale introduced by the coupling g, then each additional vertex in the perturbation series contributes a factor $(M/\Lambda)^{d_g}$, where Λ is the intrinsic energy scale of the theory and appears for dimensional reasons. If $d_g \geq 0$, these diagrams will be suppressed in the UV ($\Lambda \to \infty$). If it is negative, they will become more and more relevant and we will find divergences with higher and higher orders.[8]

Renormalizability. The renormalizability of a quantum field theory can be determined from dimensional arguments. Consider ϕ^p theory in d dimensions:

$$S = -\int d^dx \left[\frac{1}{2}\,\varphi\left(\Box + m^2\right)\varphi + \frac{g}{p!}\,\varphi^p\right]. \tag{1.77}$$

The action must be dimensionless, hence the Lagrangian has mass dimension d. From the kinetic term we read off the mass dimension of the field, namely $(d-2)/2$. The mass dimension of φ^p is thus $p\,(d-2)/2$, so that the dimension of the coupling constant must be $d_g = d + p - pd/2$.

On the other hand, a given 1PI $n-$point function $\Gamma^n(x_1, \ldots x_n)$ has mass dimension $d_\Gamma = d + n - nd/2$. This is so because the full 1PI function must have the same mass dimension as its tree-level counterpart, and the latter can be read off from the interaction terms in the Lagrangian after removing the fields (here: remove φ^p). Therefore we have in four dimensions: $d_g = 4-p$ and $d_\Gamma = 4-n$. Analogously, Eqs. (1.10) and (1.13) entail for QCD in four dimensions that the quark fields carry mass dimension 3/2 and the gluon fields dimension 1; the coupling g, the quark-gluon and four-gluon vertices are dimensionless, and the three-gluon vertex has dimension 1.

Now consider the perturbative expansion of a 1PI function in φ^p theory. Its mass dimension is fixed: $d_\Gamma = d+n-nd/2$. At a given order in perturbation theory, d_Γ can also be determined by counting the number of internal loops L (each comes with mass dimension d), the number of internal propagators I (each with dimension -2), and the number of vertices V (each with dimension d_g):

$$d_\Gamma = \underbrace{dL - 2I}_{=:D} + d_g V \qquad \Rightarrow \qquad D = d_\Gamma - d_g V\,. \tag{1.78}$$

[8] On the other hand this also means that, as long as we are only interested in $\Lambda \ll M$, non-renormalizable theories are perfectly acceptable low-energy theories. For example, chiral perturbation theory is a non-renormalizable low-energy expansion of QCD; the non-renormalizable Fermi theory of weak interactions is the low-energy limit of the electroweak theory.

D is the degree of divergence of the perturbative diagram: at higher and higher orders, Γ will contain more internal loops. If the momentum powers from the loop integration are overwhelmed by those in the denominators of the propagators, the diagram will converge in the ultraviolet ($D < 0$); if this is not the case, it will diverge ($D \geq 0$). The second equation in (1.78) states that depending on the dimension d_g of the coupling, the degree of divergence of a certain n-point function grows (or falls) with the number of internal vertices, i.e., with the order in perturbation theory.

Renormalizability means that only a *finite* number of Green functions have $D \geq 0$. If this were not the case, we would have to renormalize *every* 1PI function of the theory, which means that we would need infinitely many renormalization constants and the theory would lose predictability. This is why it is usually said that non-renormalizable theories cannot describe fundamental interactions since they are not applicable at all scales.[9] From Eq. (1.78), renormalizability implies $d_g \geq 0$, i.e., the coupling must be either dimensionless or have a positive mass dimension. For a renormalizable theory, going to higher orders does not increase the degree of divergence of an n-point function. (For a super-renormalizable theory, defined by $d_g > 0$, divergences only appear in the lowest orders of pertubation theory which is even better.) A renormalizable quantum field theory contains only a small number of superficially divergent amplitudes, namely those with a tree-level counterpart in the Lagrangian, and therefore needs only a finite number of renormalization constants.

Renormalization in φ^4. We will use φ^4 theory again as our generic example; the generalization to QCD will be straightforward. To start with, we reinterpret the fields, coupling and masses in the Lagrangian (1.77) as 'bare', unphysical quantities φ_{B}, g_{B} and m_{B}. They are related to the renormalized quantities φ, g and m via renormalization constants:

$$\varphi_{\mathrm{B}} = Z_\varphi^{1/2}\,\varphi\,, \qquad g_{\mathrm{B}} = Z_g\,g\,, \qquad m_{\mathrm{B}}^2 = Z_m\,m^2\,. \tag{1.79}$$

If we insert this in the Lagrangian, we obtain

$$\mathcal{L} = -\frac{1}{2}\,Z_\varphi\,\varphi\,(\Box + Z_m\,m^2)\,\varphi - Z_g\,Z_\varphi^2\,\frac{g}{4!}\,\varphi^4\,. \tag{1.80}$$

Consequently, when we define 1PI Green functions from the derivative of the resulting effective action with respect to the renormalized field φ, the tree-level terms will pick up a dependence on the renormalization constants. The inverse tree-level propagator and tree-level vertex are given by (let's ignore awkward factors of i):

$$\Delta_0^{-1}(p) = Z_\varphi\,(p^2 - Z_m\,m^2)\,, \qquad \Gamma_0 = Z_g\,Z_\varphi^2\,g\,. \tag{1.81}$$

In total, we have three potentially divergent renormalization constants; we will fix them by imposing three renormalization conditions. Higher n-point vertices do not introduce any new divergences; it is sufficient to determine Z_φ, Z_g and Z_m.

[9] One should however keep in mind that the renormalizability arguments developed here are based on perturbation theory. In principle, a non-renormalizable theory could also 'renormalize itself' in a nonperturbative way by developing nontrivial fixed points, which leads to the concept of *asymptotic safety*.

Now consider the Dyson-Schwinger equation for the renormalized inverse propagator from Eq. (1.52) (or, equivalently, its perturbation series). It has the generic form

$$\Delta^{-1}(p) = Z_\varphi \left(p^2 - Z_m m^2\right) + \Sigma(p)\,. \tag{1.82}$$

$\Sigma(p)$ is the self-energy; if we work in perturbation theory, its lowest-order contribution is the tadpole graph, followed by the two-loop diagram and so on. These loop diagrams will depend on internal tree-level propagators $\Delta_0(k)$ as well as tree-level vertices Γ_0. $\Sigma(p)$ has divergences which we want to eliminate. Therefore, we first have to regularize it in a suitable way. If we use dimensional regularization (in $d = 4 - \epsilon$ dimensions), we will get terms $\sim 1/\epsilon$ that diverge as $\epsilon \to 0$. For dimensional reasons the regularization procedure also introduces a scale Λ. Take for example the tadpole loop at 1-loop order: [10]

$$\begin{aligned}\int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2 - m_\mathrm{B}^2} &= \frac{m_\mathrm{B}^2}{(4\pi)^2}\,\Gamma\left(\tfrac{\epsilon}{2} - 1\right)\frac{1}{\Lambda^\epsilon}\left(\frac{m_\mathrm{B}^2}{4\pi\Lambda^2}\right)^{-\epsilon/2} \\ &\xrightarrow{\epsilon\to 0} \frac{m_\mathrm{B}^2}{(4\pi)^2}\left(-\frac{2}{\epsilon} + (\gamma - 1) + \ln\frac{m_\mathrm{B}^2}{4\pi\Lambda^2} + \mathcal{O}(\epsilon)\right).\end{aligned} \tag{1.83}$$

In $d \neq 4$ dimensions the integral, after dividing by m_B^2, is no longer dimensionless but goes like $1/\Lambda^\epsilon$. The dimension compensates that of the coupling g that would appear in front of the integral and also has dimension $4 - d = \epsilon$. However, even for $\epsilon \to 0$, the scale Λ in the denominator of the logarithm remains. Therefore, $\Sigma(p)$ generally depends not only on the momentum p, but also on the scale Λ and the parameter $\epsilon \to 0$ which produces the divergence.[11] The tadpole diagram above is actually momentum-independent; higher-order diagrams won't be. Still, the generic features of these higher-loop contributions can be read off already from Eq. (1.83).

To ensure that $\Delta^{-1}(p)$ is finite, we impose now two renormalization conditions. In combination with the third one in Eq. (1.88), those will fix the renormalization constants Z_φ and Z_m. At some arbitrary renormalization scale $p^2 = \mu^2$, we demand that the propagator reduces to that of a free particle:

$$\Delta^{-1}(p)\big|_{p^2=\mu^2} \stackrel{!}{=} p^2 - m^2\,, \qquad \frac{d}{dp^2}\,\Delta^{-1}(p)\big|_{p^2=\mu^2} \stackrel{!}{=} 1\,. \tag{1.84}$$

If we compare this with Eq. (1.82) and denote the self-energy and its derivative at the renormalization scale by $\Sigma(\mu)$ and $\Sigma'(\mu)$, respectively, we get:

$$Z_\varphi = 1 - \Sigma'(\mu)\,, \qquad Z_m = \frac{m^2 + \Sigma(\mu) - \Sigma'(\mu)\,\mu^2}{m^2\left(1 - \Sigma'(\mu)\right)}\,. \tag{1.85}$$

The renormalization 'constants' depend now on μ, Λ and ϵ and are therefore divergent. They are calculable order by order in perturbation theory. On the other hand, if we

[10] See e.g. Peskin-Schroeder, p. 249 ff for a derivation. $\Gamma(n)$ is here the Gamma function and diverges for $n = -1$; γ is the Euler-Mascheroni constant.

[11] Regularization always introduces a scale: had we used a hard cutoff instead of dimensional regularization, we would have arrived at a similar formula, where Λ would be the cutoff scale that has to be taken to infintiy (instead of $\epsilon \to 0$). Pauli-Villars regularization would introduce a regulator mass, lattice regularization an inverse lattice spacing $1/a$, etc.

put this back into Eq. (1.82), the final result for the inverse propagator is finite because the $\epsilon-$dependencies cancel:

$$\Delta^{-1}(p,\mu) = p^2 - m^2 + \Sigma(p) - \Sigma(\mu) - \Sigma'(\mu)(p^2 - \mu^2)\,. \tag{1.86}$$

For the tadpole term (1.83) this observation is trivial because $\Sigma(p)$ is momentum-independent, so that at one-loop order everything except $p^2 - m^2$ cancels on the right-hand side. At higher loops, one can check the independence of ϵ explicitly at each order. The dependence on the scale Λ cancels as well; this will be induced by differences of logarithms of the form $\ln(p^2/\Lambda^2) - \ln(\mu^2/\Lambda^2) = \ln(p^2/\mu^2)$. In turn, the resulting propagator has now picked up a dependence on the renormalization scale μ stemming from $\Sigma(\mu)$ and $\Sigma'(\mu)$, hence we write it as $\Delta(p,\mu)$.

The same analysis can be repeated for the four-point vertex which we temporarily call Γ. It depends now on three independent momenta, and its DSE (and perturbation series) reads

$$\Gamma(\{p_i\}) = Z_g\, Z_\varphi^2\, g + \Omega(\{p_i\})\,. \tag{1.87}$$

$\Omega(\{p_i\})$ collects the divergent loop contributions, and its regularized version depends again on Λ and ϵ. To ensure that the vertex is finite, we impose our third renormalization condition:

$$\Gamma(\{p_i\})\big|_{p_i^2=\mu^2} \stackrel{!}{=} g\,, \tag{1.88}$$

i.e., the vertex reduces to a free one if all momenta are evaluated at the renormalization scale. The combination of (1.87) and (1.88) fixes the remaining renormalization constant of the coupling, Z_g:

$$Z_g\, Z_\varphi^2 = 1 - \frac{\Omega(\mu)}{g} \qquad \Rightarrow \qquad \Gamma(\{p_i\},\mu) = g + \Omega(\{p_i\}) - \Omega(\mu)\,. \tag{1.89}$$

Again, Z_g depends on μ, Λ and ϵ and diverges, but the vertex remains finite because the dependence on ϵ (and also that on Λ) cancels. As a consequence, the vertex depends now on the renormalization scale μ.

It is customary to write the renormalization constants as

$$Z_\varphi = 1 + \delta Z_\varphi\,, \qquad Z_m = \frac{1}{Z_\varphi}\left(1 + \frac{\delta m^2}{m^2}\right), \qquad Z_g = \frac{1}{Z_\varphi^2}\left(1 + \frac{\delta g}{g}\right), \tag{1.90}$$

so that the Lagrangian (1.80) can be split into a piece that depends only on renormalized quantities and a counterterm that includes the new 'renormalization constants':

$$\mathcal{L} = -\frac{1}{2}\,\varphi\,(\Box + m^2)\,\varphi - \frac{g}{4!}\,\varphi^4 - \frac{1}{2}\,\varphi\,(\delta Z_\varphi\,\Box + \delta m^2)\,\varphi - \frac{\delta g}{4!}\,\varphi^4\,. \tag{1.91}$$

The counterterms will produce new tree-level propagators and vertices. Expressed in terms of δZ_φ, δm^2 and δg, the relations (1.82) and (1.87) become

$$\Delta^{-1}(p) = p^2 - m^2 + \Sigma(p) + \delta Z_\varphi\, p^2 - \delta m^2\,, \qquad \Gamma(\{p_i\}) = g + \Omega(\{p_i\}) + \delta g\,, \tag{1.92}$$

i.e., the new renormalization constants can be directly identified with the counterterms that cancel the singularities. If we apply our earlier renormalization conditions, we obtain

$$\delta Z_\varphi = -\Sigma'(\mu)\,, \qquad \delta m^2 = \Sigma(\mu) - \Sigma'(\mu)\,\mu^2\,, \qquad \delta g = -\Omega(\mu)\,. \tag{1.93}$$

Renormalization schemes. Imposing *overall* renormalization conditions of the form (1.84) and (1.88) on the Green functions defines a momentum subtraction (MOM) scheme. This is very convenient for nonperturbative calculations since at no point in the previous discussion we *needed* to resort to a perturbative expansion. Alternatively, one can also explicitly subtract only the divergent terms order by order in perturbation theory, such as the one $\sim 1/\epsilon$ in Eq. (1.83), which defines the MS scheme (minimal subtraction). In that case our definition of the renormalization scale μ is no longer available; instead, the scale $\Lambda \equiv \mu$ takes its place as it doesn't get cancelled by the subtraction anymore. (In the MOM scheme, we have essentially traded the dependence on Λ by a dependence on μ.) Another possibility is to subtract not only the divergences but *all* terms that are not explicitly dependent on $\Lambda \equiv \mu$; this defines the $\overline{\text{MS}}$ scheme (modified minimal subtraction).

In any case, all n-point functions $\Delta(p,\mu)$, $\Gamma(p,\mu)$ etc., as well as the renormalized coupling $g(\mu)$ and mass $m(\mu)$, are now defined at the renormalization scale μ. This is an unavoidable consequence of the regularization and renormalization procedure. μ is arbitrary: for example, in an onshell renormalization scheme, we would simply identify $\mu = m$ with the measurable mass of a particle. The two renormalization conditions in (1.84) would then fix the pole position and the residue of the particle's propagator. In QCD this is not possible since there are no free quarks and gluons around, and consequently there is no 'natural' scale at which we could compare these quantities with experiment. Consequently, μ remains arbitrary. In turn, the invariance of measurable quantities under a change of μ (and also different choices of regularization methods and renormalization schemes) is ensured by the renormalization group.

Renormalization in QCD. We can carry over the same analysis from φ^4 theory to QCD. QCD is a renormalizable quantum field theory because its coupling g is dimensionless. We have now several distinct fields in the Lagrangian, defined by (1.13) plus the gauge-fixing part in (1.70), which we reinterpret as 'bare' quantities. Their relationship with the renormalized quantities introduces renormalization constants:

$$\psi_{\rm B} = Z_2^{1/2}\,\psi\,, \quad A_{\rm B} = Z_3^{1/2} A\,, \quad c_{\rm B} = \widetilde{Z}_3^{1/2} c\,, \quad m_{\rm B} = Z_m\, m\,, \quad g_{\rm B} = Z_g\, g\,. \tag{1.94}$$

The naming scheme is a bit confusing but a widely used convention. To start with, we would equip each piece in the Lagrangian with its own renormalization constant, i.e., we would also have Z-terms for the quark-gluon vertex (Z_{1F}), the three-gluon vertex (Z_1), the four-gluon vertex (Z_4), the ghost-gluon vertex ($\widetilde{Z}_1$) and the term including the gauge parameter (Z_ξ). From Eqs. (1.10), (1.16) and (1.70), the resulting Lagrangian would read explicitly (modulo partial integrations): [12]

$$\begin{aligned}\mathcal{L}_{QCD} = {} & Z_2\,\bar\psi\,(i\partial\!\!\!/ - Z_m\,\mathsf{M})\,\psi + Z_{1F}\,g\,\bar\psi\,A\!\!\!/\,\psi + Z_3\,\frac{1}{2}\,A^a_\mu\,(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu)\,A^a_\nu \\ & - Z_1\,\frac{g}{2}\,f_{abc}\,(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)\,A^b_\mu\,A^c_\nu - Z_4\,\frac{g^2}{4}\,f_{abe}f_{cde}\,A^\mu_a\,A^\nu_b\,A^c_\mu\,A^d_\nu \\ & + Z_\xi\,\frac{A^a_\mu\,\partial^\mu\partial^\nu A^a_\nu}{2\xi} + \widetilde{Z}_3\,\bar c_a\,\Box\,c_a + \widetilde{Z}_1\,ig\,(\partial_\mu\,\bar c_a)\,[A^\mu, c]_a\,.\end{aligned} \tag{1.95}$$

[12] We have also rescaled the ghost fields to get rid of the coupling g in the denominator.

Fortunately, it is a consequence of gauge invariance (i.e., QCD's Slavnov-Taylor identities) that these additional renormalization constants are not independent but entirely determined by those in Eq. (1.94):

$$Z_{1F} = Z_g\, Z_2\, Z_3^{1/2}, \qquad Z_1 = Z_g\, Z_3^{3/2}, \qquad Z_4 = Z_g^2\, Z_3^2\,, \qquad \widetilde{Z}_1 = Z_g\, Z_3^{1/2}\, \widetilde{Z}_3\,. \tag{1.96}$$

In other words, it is sufficient to start from the bare Lagrangian and insert the relations (1.94) for the bare fields $\psi_{\rm B}$, $A_{\rm B}$, $c_{\rm B}$, the bare mass $m_{\rm B}$ and the bare coupling $g_{\rm B}$.[13] In summary, we only need to employ five renormalization conditions to fix the five renormalization constants in Eq. (1.94) and remove all divergences from the theory. For example, we can choose two conditions for the quark propagator, similar to those in (1.84); one condition for the gluon propagator, one for the ghost propagator, and one for the quark-gluon vertex analogous to that in (1.88). All renormalization constants depend on μ and Λ (and in dimensional regularization on $\epsilon \to 0$); the renormalized Green functions, the renormalized quark mass $m(\mu)$ and the renormalized coupling $g(\mu)$ depend on the renormalization scale μ only: 'the theory is defined at the scale μ'.

Anomalous breaking of scale invariance. The fact that renormalization introduces a scale has interesting consequences. Take for example QCD with massless quarks: its classical Lagrangian has no intrinsic scale; it is scale invariant. If we were to compute the hadron spectrum of massless QCD, we would expect all hadrons to be massless as well since there is nothing to set the scale. This can't be right, and indeed it is no longer true at the quantum level because regularization and renormalization breaks scale invariance by introducing a scale in the theory ('anomalous breaking of scale invariance'). Therefore, a renormalized quantum field theory is usually quoted together with a scale that sets the units of mass and has to be determined experimentally. The renormalization group will tell us how to trade the arbitrary scale μ for a scale $\Lambda_{\rm QCD}$ that we can relate to experiment.

Callan-Symanzik equation. From Eqs. (1.79) and (1.94) we can read off how a renormalized 1PI Green function ($\Gamma^n = \delta^n\Gamma/\delta\varphi^n$) with n legs, corresponding to a field φ or several different fields, is related to its bare counterpart ($\Gamma^n_{\rm B} = \delta^n\Gamma/\delta\varphi^n_{\rm B}$):

$$\Gamma^n(\{p_i\}, g, m, \mu) = Z_\varphi^{n/2}\, \Gamma^n_{\rm B}(\{p_i\}, g_{\rm B}, m_{\rm B})\,. \tag{1.97}$$

The bare quantities cannot depend on the renormalization scale μ. If we apply the derivative $\mu\, d/d\mu$ under the constraint $d\Gamma^n_{\rm B}/d\mu = 0$, we obtain the Callan-Symanzik equation:

$$\left(\mu\,\frac{\partial}{\partial\mu} + \underbrace{\mu\,\frac{dg}{d\mu}}_{\beta(g)}\,\frac{\partial}{\partial g} + \underbrace{\mu\,\frac{dm}{d\mu}}_{m\gamma_m(g)}\,\frac{\partial}{\partial m}\right)\Gamma^n = n\,\underbrace{\frac{\mu}{2}\,\frac{d\ln Z_\varphi}{d\mu}}_{\gamma(g)}\,\Gamma^n\,. \tag{1.98}$$

Here we defined the β function $\beta(g)$, the anomalous mass dimension $\gamma_m(g)$, and the anomalous dimension of the field $\gamma(g)$ which determine the change of the coupling, the mass and the field renormalization under a change of the renormalization scale.

[13] Another consequence of gauge invariance is $Z_\xi = 1$, which entails that the longitudinal part of the gluon propagator stays unrenormalized.

For Green functions that depend on more than one field we would have to include a separate $\gamma(g)$ for each of them.

The Callan-Symanzik equation entails that a shift of the renormalization scale can be compensated by an appropriate shift of the coupling, the mass and the fields. Suppose for the moment that $\gamma(g) = 0$, so that Z_φ is independent of μ. We also set $m = 0$ to simplify the discussion. The l.h.s of the equation then implies $d\Gamma^n/d\mu = 0$, i.e. also the renormalized n-point function is $\mu-$independent. A change of the renormalization point can then always be compensated by a shift of the coupling:

$$\Gamma^n(\{p_i\}, g(\mu), \mu) = \Gamma^n(\{p_i\}, g(\mu_0), \mu_0)\,. \tag{1.99}$$

More importantly, the Callan-Symanzik equation also allows us to compensate the *momentum* dependence of a Green function by a change in its coupling. Consider a Green function with mass dimension D; it can be written as

$$\Gamma^n(\{p_i\}, g(\mu), \mu) = \mu^D f\left(\left\{\frac{p_i}{\mu}\right\}, g(\mu)\right) = \mu_0^D\, f\left(\left\{\frac{p_i}{\mu_0}\right\}, g(\mu_0)\right), \tag{1.100}$$

where the function f is dimensionless. The first equality is simply a dimensional argument, the second follows from Eq. (1.99) since the expression is independent of μ. Now replace all momenta $p_i \to \lambda p_i$, where $\lambda = \mu/\mu_0$:

$$f\left(\lambda\left\{\frac{p_i}{\mu_0}\right\}, g(\mu_0)\right) = \lambda^D f\left(\left\{\frac{p_i}{\mu_0}\right\}, g(\lambda\mu_0)\right). \tag{1.101}$$

Hence, at a fixed renormalization point μ_0, a uniform rescaling of momenta can be compensated by an according shift of the coupling on which the Green function depends.[14] This is the origin of the momentum-dependent 'running coupling' that will allow us to extract information about the large-momentum behavior of QCD from high-energy scattering experiments. Roughly speaking, instead of computing the actual momentum dependence of the various Green functions that enter in scattering amplitudes, it is sufficient to keep their tree-level terms and replace all instances of the coupling g by its momentum-dependent version $g(Q^2)$. This simplifies matters considerably since the running of the coupling can be computed in perturbation theory, which we will do in the following.

Running coupling and β function. Suppose we want to switch from one momentum scale (μ_0) to another (μ), and express this change in terms of the variable $t := \ln(\mu/\mu_0) \in [-\infty, \infty]$. This will also simplify the formulas a bit: for fixed μ_0, we have $\mu\, d/d\mu = d/dt$. Since g is dimensionless, it will depend on the dimensionless quantity t. Let's denote the original coupling g, defined at the scale μ_0, by $g(0)$ and the new one by $g(t)$. The β function tells us how the coupling constant evolves from μ_0 to μ:

$$\beta(g) = \frac{dg(t)}{dt} \;\Rightarrow\; \int_{g(0)}^{g(t)} \frac{dg}{\beta(g)} = \int_0^t dt' = t\,, \tag{1.102}$$

[14] If we reversed our simplifications $\gamma(g) = 0$ and $m = 0$, the equation would pick up a scaling factor that depends on $\gamma(g)$, and the renormalized mass would obtain a scaling factor $\sim \gamma_m(g)$, hence the name 'anomalous dimensions'.

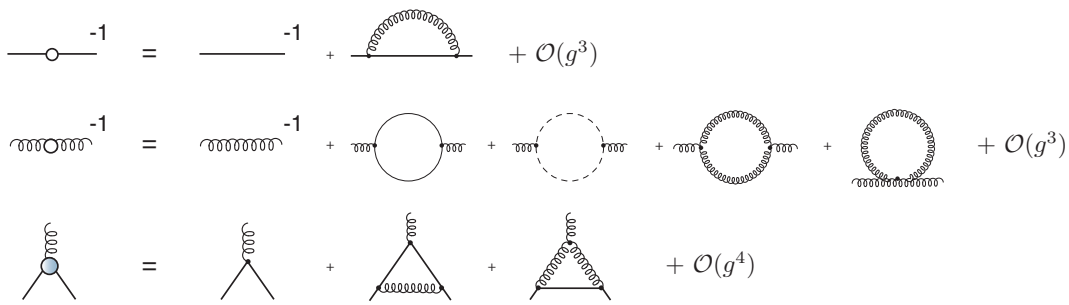

FIGURE 1.5: Diagrams that contribute to Z_2, Z_3 and Z_{1F} up to $\mathcal{O}(g^2)$, i.e., at 1-loop.

which we can solve to obtain the $t-$dependence of the coupling $g(t)$. In order to do so, we must first calculate Z_g. From Eq. (1.94) we have:

$$g = \frac{g_{\rm B}}{\mu^{\epsilon/2}\, Z_g} \quad \Rightarrow \quad \beta(g) = -\left(\frac{\epsilon}{2} + \frac{d}{dt}\ln Z_g\right) g\,. \tag{1.103}$$

In $d = 4 - \epsilon \neq 4$ dimension, the coupling becomes dimensionful; in QCD its dimension is $\epsilon/2$. We have defined g above so that it stays dimensionless in arbitrary dimensions.

From Eqs. (1.95)–(1.96) we see that Z_g in the Lagrangian always appears in combination with other renormalization constants, so we can calculate it from any of these relations for the quark-gluon, three-gluon, four-gluon or ghost-gluon vertices. The most common procedure is the first one, hence we must determine Z_2, Z_3 and Z_{1F} and combine them in the end to obtain Z_g. At one-loop order they correspond to the counterterms in the perturbation series for the quark propagator, the gluon propagator and the quark-gluon vertex, cf. Fig. 1.5. If we use dimensional regularization and the MS scheme, the counterterms cancel only the $1/\epsilon$ singularities and nothing else; hence the $\mu-$dependence in Z_g comes entirely from its dependence on g itself. The calculation is a bit lenghty and we only state the result here (to 1-loop):

$$Z_g = 1 - \frac{b}{\epsilon}\, g^2 + \ldots \;\Rightarrow\; \frac{d\ln Z_g}{dt} = -\frac{2b}{\epsilon}\, g\, \beta(g) + \ldots \tag{1.104}$$

where $b = \beta_0/(4\pi)^2$ and $\beta_0 = 11 - \frac{2}{3}\, N_f$ is a constant that depends on the number of flavors. Inserting this in Eq. (1.103) yields

$$\beta(g) = -\frac{\epsilon g}{2} - b\, g^3 + \ldots \;\xrightarrow{\varepsilon\to 0}\; -b\, g^3 + \ldots\,, \tag{1.105}$$

with higher-order terms $\sim g^5$, g^7, etc. Except for the two lowest-order coefficients at $\mathcal{O}(g^3)$ and $\mathcal{O}(g^5)$, the β function depends on the renormalization scheme. Even a MOM scheme is not unique since we can choose to distribute our five renormalization conditions differently. For example, instead of imposing conditions on the quark, gluon and ghost propagators and the quark-gluon vertex, we could fix the three-gluon, four-gluon vertices etc. at the renormalization point, which would correspond to different MOM schemes. Consequently, also the running coupling $g(t)$ at higher orders will be

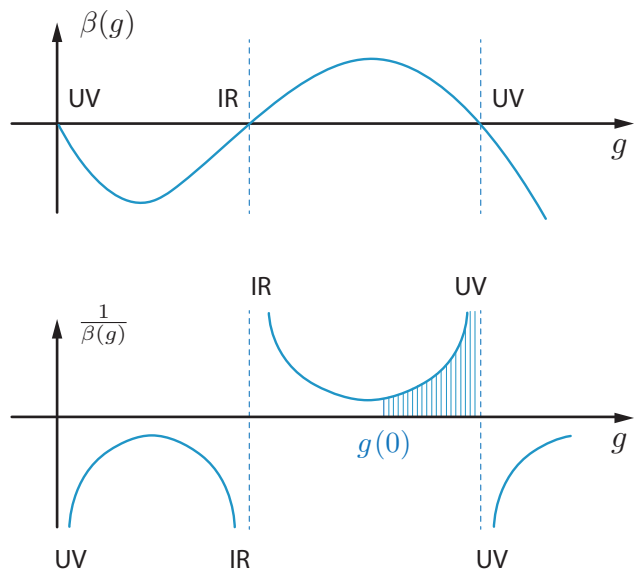

FIGURE 1.6: Possible shape of the β function and its inverse that appears in Eq. (1.102).

scheme-dependent, and the same argument can be made for the running mass $m(t)$ and the Green functions of the theory.

Nevertheless, the coefficient b in Eq. (1.105) is unique. The negative sign of the β function at $g \to 0$ has important consequences: it implies that QCD is an asymptotically free theory, i.e., the effective coupling approaches zero as the momenta approach infinity. For general theories, the zeros $\beta(g^\star) = 0$ play a special role. The respective values of $g^\star$ are fixed points under a renormalization-group evolution because the coupling in the vicinity of $g^\star$ does not change under $t-$evolution ($dg/dt = 0$). From Eq. (1.102) one infers that for $t \to \pm\infty$ the l.h.s. must diverge: this happens when $g(t)$ approaches the fixed point nearest to $g(0)$, or when it goes to infinity because there is no zero of $\beta(g)$ to approach (see Fig. 1.6). Whether the fixed point corresponds to $t \to \infty$ or $t \to -\infty$ depends on the sign of the $\beta-$function and the integration direction. For example, $t \to +\infty$ implies $g(t) > g(0)$ and $\beta > 0$, or $g(t) < g(0)$ and $\beta < 0$ (again, see Fig. 1.6). The nature of the fixed point is thus determined by the sign of the derivative $\beta'(g)$ at $g = g^\star$:

- ultraviolet fixed point: $\beta'(g^\star) < 0 \;\Leftrightarrow\; g^\star = g(t \to \infty)$
- infrared fixed point: $\beta'(g^\star) > 0 \;\Leftrightarrow\; g^\star = g(t \to -\infty)$

The origin $g = 0$ is always a fixed point since $\beta(0) = 0$. A theory is asymptotically free if $g = 0$ is a UV fixed point, since then the coupling becomes small for $t \to \infty$ (as for example in QCD). It is infrared stable if $g = 0$ is an IR fixed point (examples: QED, φ^4 theory). If there is only one coupling in the system (and if we set $m = 0$, so that the 'theory space' is one-dimensional), then the domains separated by fixed points correspond to different theories, or different phases of the same theory.

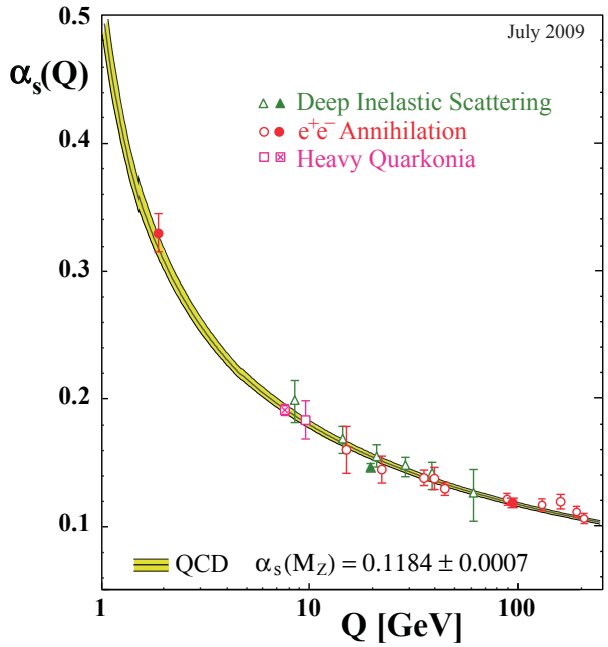

FIGURE 1.7: Overview of α_s measurements, from Bethke, Eur. Phys. J. C64 (2009).

For QCD this implies that $g(t)$ is small at large momenta: there, quarks and gluons behave as asymptotically free particles and we can apply perturbation theory. On the other hand, it also means that the coupling increases at small momenta and perturbation theory will eventually fail. In that region nonperturbative effects related to the formation of bound states become important. The opposite case is QED, where $\beta(g \to 0)$ is positive, i.e., the coupling grows with increasing momenta. It actually grows very slowly, so that perturbation theory works very well over many orders of magnitude. Due to the N_f dependence in β_0 (below Eq. (1.104)), the QCD β function is negative only as long as $N_f \leq 16$; for a larger number of flavors we would lose asymptotic freedom.

Now let's return to Eq. (1.105). If we put the resulting $\beta-$function into (1.102) and solve the equation for $g(t)$, we obtain the 1-loop result for the running coupling:

$$g(t)^2 = \frac{g(0)^2}{1 + 2b\, t g(0)^2} \qquad \text{for} \qquad t \to \infty\,. \tag{1.106}$$

We can trade the dependence on μ_0 by a dependence on a scale $\Lambda_{\rm QCD}$ if we define

$$g(0)^2 := \frac{1}{b\,\ln(\mu_0^2/\Lambda_{\rm QCD}^2)} \quad \Rightarrow \quad g(t)^2 = \frac{1}{b\,\ln(\mu^2/\Lambda_{\rm QCD}^2)}\,. \tag{1.107}$$

$\Lambda_{\rm QCD}$ marks the scale where perturbation theory definitely breaks down since it produces an unphysical Landau pole in the perturbative expansion. $\Lambda_{\rm QCD}$ sets the unit of the mass scale, and all dimensionful quantities are expressed in terms of this scale parameter. It depends not only on the order in perturbation theory, but also on the

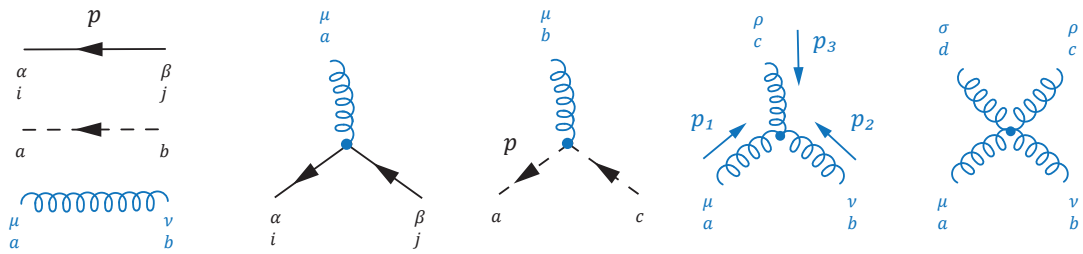

FIGURE 1.8: Tree-level propagators and vertices in the QCD action.

renormalization scheme, and on the number of active flavors at the scale where the coupling is probed (due to the N_f dependence in β_0). Usually the running coupling is written as $\alpha_s(t) := g(t)^2/(4\pi)$; its knowledge is sufficient to determine much of the behavior of high-energy scattering. Comparison of $\alpha_s(t)$ at four-loop order with experimental results yields the value $\Lambda_{\overline{\text{MS}}}^{N_f=5} = 213$ MeV (same reference as in Fig. 1.7).

Feynman rules for QCD. We have now everything in place to write down the final expressions for the renormalized tree-level propagators and vertices in QCD. They are necessary for perturbative calculations since the dressed n-point functions at large momenta revert to these forms, but they also enter as inputs for nonperturbative studies. The Feynman rules for the quark, gluon and ghost propagator are given by

$$S_0(p) = \frac{i}{Z_2}\,\frac{\not{p} + m_0}{p^2 - m_0^2}\,, \quad D_0^{\mu\nu}(p) = -\frac{i}{p^2}\left(\frac{T_p^{\mu\nu}}{Z_3} + \xi\, L_p^{\mu\nu}\right), \quad G_0(p) = -\frac{i}{\widetilde{Z}_3\, p^2}\,, \tag{1.108}$$

where $m_0 = Z_m\, m$ and m is the renormalized quark mass. We abbreviated the longitudinal and transverse projectors that appear in the gluon propagator by $L_p^{\mu\nu} = p^\mu\, p^\nu/p^2$ and $T_p^{\mu\nu} = g^{\mu\nu} - L_p^{\mu\nu}$. The tree-level quark-gluon and ghost-gluon vertices read (see Fig. 1.8 for the kinematics)

$$\Gamma^\mu_{\text{q},0} = ig\, Z_{1F}\, \gamma^\mu\, \mathsf{t}_a\,, \qquad \Gamma^\mu_{\text{gh},0} = g\, \widetilde{Z}_1\, f_{abc}\, p^\mu\,, \tag{1.109}$$

and the three- and four-gluon vertices are given by

$$\begin{aligned}
\Gamma^{\mu\nu\rho}_{\text{3g},0} &= g\, Z_1\, f_{abc}\left[(p_1 - p_2)^\rho g^{\mu\nu} + (p_2 - p_3)^\mu g^{\nu\rho} + (p_3 - p_1)^\nu g^{\rho\mu}\right], \\
\Gamma^{\mu\nu\rho\sigma}_{\text{4g},0} &= -ig^2\, Z_4\left[f_{abe} f_{cde}\,(g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}) + f_{ace} f_{bde}\,(g^{\mu\nu} g^{\rho\sigma} - g^{\nu\rho} g^{\mu\sigma})\right. \\
&\qquad \left. + f_{ade} f_{cbe}\,(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma})\right].
\end{aligned} \tag{1.110}$$

The vertex renormalization constants are related to Z_m, Z_g, Z_2, Z_3 and $\widetilde{Z}_3$ via Eq. (1.96).