Chapter 2

Hadrons

In the last chapter we essentially ignored the flavor structure of QCD because it was less relevant for the properties of quarks and gluons compared to their color structure. Here we will turn the wheel around and focus exclusively on the flavor symmetries. QCD's local gauge symmetry doesn't tell us much about the systematics of the hadron spectrum other than their color-singlet nature. This allows us to construct $q\bar{q}$ (mesons), qqq (baryons), and in principle also more complicated states. On the other hand, the global flavor symmetries of QCD become now important and introduce new effects that are observable (or conspicuously missing) in the mass spectrum, for example: the multiplet structure, spontaneous chiral symmetry breaking or the $U_A(1)$ anomaly.

2.1 Flavor symmetries and currents

Noether theorem. Any continuous (local or global) symmetry transformation which leaves the action invariant implies the existence of a conserved current, where the corresponding charge is a constant of motion. Let's exemplify the Noether theorem for a generic field theory with action $S = \int d^4x\, \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$. Consider a global transformation $\varphi_i' = (e^{i\varepsilon_a \mathsf{t}_a})_{ij}\, \varphi_j = \varphi_i + \delta\varphi_i$ of the fields with generators t_a, satisfying $[\mathsf{t}_a, \mathsf{t}_b] = i f_{abc}\, \mathsf{t}_c$. Compute the variation of the action with respect to the group parameter ε_a for solutions of the classical equations of motion:

$$\begin{aligned}\delta S &= \int\limits_V d^4x\, \delta\mathcal{L} = \int\limits_V d^4x \sum_i \left[\frac{\partial \mathcal{L}}{\partial \varphi_i}\, \delta\varphi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\, \delta(\partial_\mu \varphi_i)\right] \\ &= \int\limits_V d^4x \left[\partial_\mu \underbrace{\left(\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\, \delta\varphi_i\right)}_{=:\, -\varepsilon_a\, j_a^\mu} + \sum_i \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\right)}_{=\,0 \text{ for classical solutions}} \delta\varphi_i\right].\end{aligned} \tag{2.1}$$

The second bracket vanishes for solutions of the Euler-Lagrange equations. Hence, if the classical action is invariant, there is one conserved current for each generator of the symmetry group when evaluated along the classical trajectories:

$$\partial_\mu\, j_a^\mu = -\frac{\delta \mathcal{L}}{\delta \epsilon_a} = 0\,. \tag{2.2}$$

We can further use Gauss' law to convert the volume to a surface integral. As long as the surface term at the spatial boundary is zero, this yields a conserved charge for each generator t_a of the symmetry group:

$$\frac{\delta S}{\delta \varepsilon_a} = -\int_{\partial V} d\sigma_\mu \, j^\mu_\alpha = 0 \quad \Rightarrow \quad Q_a(t) = \int d^3x \, j^0_a(x) = \text{const.} \tag{2.3}$$

If the symmetry is broken classically and the action is not invariant, or if the fields do not obey the equations of motion, we can still *define* currents and corresponding charges but they will not be conserved.

When the classical field theory is quantized, the fields $\varphi_i(x)$ and charges $Q_a(t)$ become operators on the state space of the theory. It follows from the equal-time (anti-) commutation relations of the fields that the charges satisfy the same commutator relations as the generators of the symmetry group,

$$[Q_a, Q_b] = i f_{abc} \, Q_c \,, \tag{2.4}$$

and thereby form a representation of the Lie algebra on the Hilbert space, the so-called charge algebra. This is true even if the charges are time-dependent, i.e., if the symmetry is broken. The Heisenberg equations of motion,

$$\frac{dQ_a}{dt} = i \, [H_{\text{QCD}}, Q_a] \,, \tag{2.5}$$

which are are a consequence of translation invariance and hold for any polynomial of the fields, then entail that QCD's Hamiltonian commutes with the charges as long as they are conserved. Hence, the eigenstates of the Hamiltonian with the same eigenvalue of Q_a are also mass-degenerate (if they share all other quantum numbers such as angular momentum, parity etc. as well). The spectrum of the theory can then be labeled in terms of the irreducible representations of the symmetry group.

There are also other possibilities how symmetries can be broken:

- **Spontaneous symmetry breaking:** This can happen when the dynamics of the field contains (massless) long-range interactions. The classical action is still invariant and the current is conserved, but the spatial integral and the charge in Eq. (2.3) are then no longer well-defined. The vacuum, the quantum effective action and the Green functions of the theory do no longer share the global symmetry of the Lagrangian. Each generator that does not leave the vacuum invariant corresponds to a massless Goldstone boson. The QCD example is chiral symmetry, or more precisely, the group $SU(N_f)_A$ for vanishing quark masses.

- **Anomalous symmetry breaking:** Here the classical action is again invariant, but the symmetry is broken at the quantum level due to renormalization. This can happen when no symmetry-preserving regulator exists. A typical candidate are again chiral symmetries: in dimensional regularization, γ_5 has no natural extension to $d \neq 4$ dimensions; a Pauli-Villars regulator breaks chiral symmetry explicitly due to a mass term, etc. In contrast to the case of spontaneous breaking, also the current is no longer conserved but picks up additional terms. We already mentioned the anomalous breaking of scale invariance; another example is the $U(1)_A$ anomaly in QCD.

Flavor Lagrangian. In order to discuss flavor symmetries, we only need to consider the quark part of the QCD Lagrangian since only the quark fields carry flavor labels:

$$\mathcal{L} = \overline{\psi}\,(i\partial\!\!\!/ - \mathsf{M})\,\psi + g\,\overline{\psi}\,A\!\!\!/\,\psi\,. \tag{2.6}$$

We will work with unrenormalized quantities for simplicity and discuss renormalization when necessary. We will also suppress the color indices of the quarks from Eq. (1.1) and denote the flavor indices instead by $i = 1\dots N_f$. The spinor fields $\overline{\psi}_{\alpha,i}(x)$, $\psi_{\alpha,i}(x)$ transform under the fundamental representation of $SU(N_f)$:

$$\psi'(x) = U\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)\,U^\dagger \qquad \text{with} \quad U = e^{i\sum_a \varepsilon_a \mathsf{t}_a}\,. \tag{2.7}$$

The t_a are now the $SU(N_f)$ generators, e.g., the Pauli matrices $\mathsf{t}_a = \tau_a/2$ for two flavors and Gell-Mann matrices $\mathsf{t}_a = \lambda_a/2$ for three flavors (see Appendix A). In the two-flavor case, the quark mass matrix in the Lagrangian has the form

$$\mathsf{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{m_u + m_d}{2}\,\mathbb{1} + (m_u - m_d)\,\mathsf{t}_3\,, \tag{2.8}$$

whereas in the three-flavor case it is given by $\mathsf{M} = \text{diag}(m_u, m_d, m_s)$ or

$$\mathsf{M} = \frac{m_u + m_d + m_s}{3}\,\mathbb{1} + (m_u - m_d)\,\mathsf{t}_3 + \frac{m_u + m_d - 2m_s}{\sqrt{3}}\,\mathsf{t}_8\,. \tag{2.9}$$

Flavor symmetries. We are interested in the properties of the Lagrangian under the global transformations $U(1)_V \times SU(N_f)_V \times SU(N_f)_A \times U(1)_A$ of the quark fields:

$$SU(N_f)_V \;\ni\; e^{i\sum_a \varepsilon_a \mathsf{t}_a} \qquad \Rightarrow \qquad \frac{\delta\psi}{\delta\varepsilon_a} = \mathsf{t}_a i\psi\,, \qquad \frac{\delta\overline{\psi}}{\delta\varepsilon_a} = -i\overline{\psi}\,\mathsf{t}_a\,, \tag{2.10}$$

$$SU(N_f)_A \;\ni\; e^{i\gamma_5\sum_a \varepsilon_a \mathsf{t}_a} \qquad \Rightarrow \qquad \frac{\delta\psi}{\delta\varepsilon_a} = \gamma_5\,\mathsf{t}_a i\psi\,, \quad \frac{\delta\overline{\psi}}{\delta\varepsilon_a} = i\overline{\psi}\,\gamma_5\,\mathsf{t}_a\,. \tag{2.11}$$

The subscripts V and A indicate that these transformations will induce vector and axialvector currents. For the corresponding flavor-singlet transformations $e^{i\varepsilon} \in U(1)_V$ and $e^{i\gamma_5\varepsilon} \in U(1)_A$, where ε is now just a number, the variations of the fields are obtained by replacing $\mathsf{t}_a \to 1$. Notice the positive sign for the $\delta\overline{\psi}$ terms in the axial case which follows from the anticommutation of γ_5 and γ_0 in obtaining $\overline{\psi} = \psi^\dagger\gamma_0$. We will also make frequent use of the following quark bilinears:

$$j^\Gamma_a(x) := \overline{\psi}(x)\,\Gamma\,\mathsf{t}_a\,\psi(x)\,, \qquad j^\Gamma(x) := \overline{\psi}(x)\,\Gamma\,\psi(x)\,, \tag{2.12}$$

where $\Gamma \in \{\gamma^\mu,\, \gamma^\mu\gamma_5,\, \mathbb{1},\, i\gamma_5\}$ are vector, axialvector, scalar and pseudoscalar Dirac matrices. We denote the corresponding (Hermitian) vector, axialvector, scalar and pseudoscalar currents $j^\Gamma_{(a)}(x)$ by [1]

$$\gamma^\mu \to V^\mu_{(a)}(x), \qquad \gamma^\mu\gamma_5 \to A^\mu_{(a)}(x), \qquad \mathbb{1} \to S_{(a)}(x), \qquad i\gamma_5 \to P_{(a)}(x). \tag{2.13}$$

In the following we'll investigate these symmetry transformations in detail.

[1] Here's a clash of notation: A^μ denotes both the axialvector current and the gluon field. Fortunately we won't be dealing with gluons for a while, and if so we will use the gluon field-strength tensor $F^{\mu\nu}$ instead. Unless stated otherwise, A^μ will refer to an axialvector current from now on.

$\mathbf{U(1)_V}$: The Lagrangian is invariant under a global phase transformation $\psi' = e^{i\varepsilon}\psi$. The corresponding flavor-singlet vector current according to Eq. (2.1) is

$$V^\mu = -\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_{\alpha,i})}\,\frac{\delta\psi_{\alpha,i}}{\delta\varepsilon} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_{\alpha,i})}\,\frac{\delta\overline{\psi}_{\alpha,i}}{\delta\varepsilon}\right] = \overline{\psi}\,\gamma^\mu\,\psi\,, \tag{2.14}$$

where we used

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\overline{\psi}\,\gamma^\mu\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0\,. \tag{2.15}$$

Current conservation $\partial_\mu V^\mu = 0$ can be verified by inserting the solutions of the classical Dirac equations of motion from Eq. (1.20):

$$\partial\!\!\!/\,\psi = (g\,A\!\!\!/ - \mathsf{M})\,i\psi\,, \qquad \overline{\psi}\,\overleftarrow{\partial\!\!\!/} = i\overline{\psi}\,(-g\,A\!\!\!/ + \mathsf{M})\,. \tag{2.16}$$

The conserved charge is

$$Q^V(t) = \int d^3x\,\overline{\psi}\,\gamma^0\,\psi = \int d^3x\,\psi^\dagger\,\psi = \text{const.} \tag{2.17}$$

and reflects fermion number conservation, since it counts the number of quarks minus antiquarks in the state. If we define $n_q = (\#q) - (\#\bar{q})$ for each flavor, then the eigenvalue of Q^V (which we also call Q^V) is the baryon number. For three flavors:

$$B := \frac{Q^V}{3} = \frac{n_u + n_d + n_s}{3}\,, \tag{2.18}$$

and the $U(1)_V$ symmetry entails baryon number conservation.

$\mathbf{SU(N_f)_V}$: is explicitly broken by the mass matrix $\mathsf{M} \neq m\,\mathbb{1}$. We can still write down the currents, one for each generator of the group, and compute their divergence:

$$V_a^\mu = \overline{\psi}\,\gamma^\mu\,\mathsf{t}_a\,\psi\,, \qquad \partial_\mu\,V_a^\mu = i\overline{\psi}\,[\mathsf{M},\mathsf{t}_a]\,\psi\,. \tag{2.19}$$

The Lagrangian is invariant only if all quark masses are identical; the $(N_f^2 - 1)$ vector currents are then conserved: $\partial_\mu\,V_a^\mu = 0$, and so are the corresponding charges:

$$Q_a^V(t) = \int d^3x\,\psi^\dagger\,\mathsf{t}_a\,\psi = \text{const.} \tag{2.20}$$

Because the diagonal generators (t_3 in the two-flavor and t_3, t_8 in the three-flavor case) commute with each other and hence also with the mass matrix, their currents are conserved even if $\mathsf{M} \neq m\,\mathbb{1}$. They define the isospin and hypercharge currents:

$$\begin{aligned} V_3^\mu &= \overline{\psi}\,\gamma^\mu\,\mathsf{t}_3\,\psi = \frac{1}{2}\left(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d\right), \\ V_8^\mu &= \overline{\psi}\,\gamma^\mu\,\mathsf{t}_8\,\psi = \frac{1}{2\sqrt{3}}\left(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d - 2\bar{s}\gamma^\mu s\right), \end{aligned} \tag{2.21}$$

and their conserved charges define the third component of the isospin $I_3 = Q_3^V$ and the hypercharge $Y = (2/\sqrt{3})\,Q_8^V$. Conservation of V_3^μ, V_8^μ and V^μ from Eq. (2.14) entails that the flavor-diagonal vector currents $\bar{u}\,\gamma^\mu u$, $\bar{d}\,\gamma^\mu d$ and $\bar{s}\,\gamma^\mu s$ are always conserved which reflects flavor conservation in QCD. From the eigenvalues of B, $I_3 = (n_u - n_d)/2$ and $Y = (n_u + n_d - 2n_s)/3$ we obtain

$$Y = B + S\,, \qquad Q = I_3 + \frac{Y}{2} = \frac{2}{3}\,n_u - \frac{1}{3}\,n_d - \frac{1}{3}\,n_s\,, \tag{2.22}$$

where $S = -n_s$ is the strangeness and Q the charge of the state. This will allow us to arrange hadrons into $\{I_3, S\}$ multiplets even if the underlying flavor symmetry is broken. The remaining flavor-changing vector currents have divergences proportional to quark-mass differences; if we go back to the two-flavor case with $m_u \neq m_d$ and use instead of $\mathsf{t}_{1,2} = \tau_{1,2}/2$ the generators

$$\mathsf{t}_+ = \mathsf{t}_1 + i\mathsf{t}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{t}_- = \mathsf{t}_1 - i\mathsf{t}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.23}$$

we obtain

$$\partial_\mu V^\mu_+ = i(m_u - m_d)\,\bar{u}d\,, \qquad \partial_\mu V^\mu_- = -i(m_u - m_d)\,\bar{d}u\,. \tag{2.24}$$

$\mathbf{SU(N_f)_A}$ **:** is explicitly broken by the mass matrix $\mathsf{M} \neq 0$:

$$A^\mu_a = \overline{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi\,, \qquad \partial_\mu A^\mu_a = i\overline{\psi}\,\{\mathsf{M}, \mathsf{t}_a\}\,\gamma_5\,\psi\,. \tag{2.25}$$

Even if all quark masses are equal, there remains a non-zero contribution proportional to the quark mass:

$$\partial_\mu A^\mu_a = 2m\,\overline{\psi}\,i\gamma_5\,\mathsf{t}_a\psi = 2mP_a\,. \tag{2.26}$$

This is the PCAC (partially conserved axialvector current) relation: the divergence of the axialvector current is a pseudoscalar density. This equation will become extremely useful later. Using (2.23) in the two-flavor case, we obtain

$$\begin{aligned} \partial_\mu A^\mu_+ &= i(m_u + m_d)\,\bar{u}\gamma_5 d\,, \\ \partial_\mu A^\mu_- &= i(m_u + m_d)\,\bar{d}\gamma_5 u\,, \\ \partial_\mu A^\mu_3 &= im_u\,\bar{u}\gamma_5 u - im_d\,\bar{d}\gamma_5 d\,, \end{aligned} \tag{2.27}$$

which are the creation operators for the three pions π^+, π^- and π^0.

On the other hand, Eq. (2.25) entails that the axial currents are conserved in the chiral limit where *all* quark masses go to zero: $\partial_\mu A^\mu_a = 0$. Then also all axial charges are conserved:

$$Q^A_a(t) = \int d^3x\,\psi^\dagger\gamma_5\,\mathsf{t}_a\,\psi = \text{const.} \tag{2.28}$$

Since the vector currents are conserved as well in that case, we have an enlarged flavor symmetry $SU(N_f)_V \times SU(N_f)_A \simeq SU(N_f)_L \times SU(N_f)_R$, namely chiral symmetry. It will turn out that the $SU(N_f)_A$ part of chiral symmetry is spontaneously broken at the quantum level; nevertheless all relations for the currents remain valid.

$\mathbf{U(1)_A}$ **:** is classically conserved for $\mathsf{M} = 0$, but not preserved after quantization which leads to the $U(1)_A$ anomaly. The divergence of the axialvector singlet current picks up an anomalous contribution whose origin and consequences we will discuss later:

$$A^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\psi\,, \qquad \partial_\mu A^\mu = 2i\,\overline{\psi}\,\mathsf{M}\,\gamma_5\,\psi + \frac{g^2 N_f}{32\pi^2}\,\widetilde{F}^{\mu\nu}_a\,F^a_{\mu\nu}\,. \tag{2.29}$$

Chiral symmetry. The enlarged $SU(N_f)_V \times SU(N_f)_A$ symmetry in the limit of vanishing quark masses is equivalent to a chiral symmetry $SU(N_f)_L \times SU(N_f)_R$. To see this, define the chiral projectors

$$\mathsf{P}_\pm := \frac{1}{2}(\mathbb{1} \pm \gamma_5) \quad \Rightarrow \quad \mathsf{P}_\omega = \mathsf{P}_\omega^\dagger, \quad \sum_\omega \mathsf{P}_\omega = \mathbb{1}, \quad \mathsf{P}_\omega^2 = \mathsf{P}_\omega, \quad \mathsf{P}_\omega \, \mathsf{P}_{-\omega} = 0\,. \tag{2.30}$$

Chirality is here denoted by the index + (right-handed) or − (left-handed). The projectors can be used to define right- and left-handed spinors:

$$\psi_\omega = \mathsf{P}_\omega \, \psi\,, \qquad \bar{\psi}_\omega = \bar{\psi}\, \mathsf{P}_{-\omega}\,, \tag{2.31}$$

where we used the property $\bar{\mathsf{P}}_\omega = \gamma_0 \, \mathsf{P}_\omega^\dagger \gamma_0 = \mathsf{P}_{-\omega}$. Because of $\mathsf{P}_{-\omega}\, \gamma^\mu = \gamma^\mu \, \mathsf{P}_\omega$, the Lagrangian (2.6) in the massless case decouples in a left-handed and right-handed part:

$$\mathcal{L} = \bar{\psi}\, i\rlap{\,/}D\, \psi = \sum_\omega \bar{\psi}_\omega \, i\rlap{\,/}D\, \psi_\omega\,. \tag{2.32}$$

It is invariant under separate $SU(N_f) \times SU(N_f)$ transformations of the left- and right-handed spinors, with independent group parameters ε_a^+ and ε_a^-:

$$\psi'_\omega = U_\omega \, \psi_\omega\,, \qquad U_\omega = e^{i\sum_a \varepsilon_a^\omega \, \mathsf{t}_a}\,, \qquad U_\omega^\dagger = U_\omega^{-1} = \bar{U}_\omega\,. \tag{2.33}$$

For example, ψ_+ transforms under the fundamental representation of the right-handed $SU(N_f)$ but as a singlet with respect to the left-handed one.

Let's see how general Dirac matrices transform under chiral symmetry. There are two possible cases:

$$(1) \quad \Gamma = \gamma^\mu,\, \gamma^\mu\gamma_5 \qquad \Rightarrow \qquad \begin{array}{l} \mathsf{P}_\omega \, \Gamma \, \mathsf{P}_\omega = 0, \\ \bar{\psi}_{-\omega}\, \Gamma \, \psi_\omega = 0 \end{array} \qquad \Rightarrow \quad \bar{\psi}\, \Gamma\, \psi = \sum_\omega \bar{\psi}_\omega \, \Gamma\, \psi_\omega\,, \tag{2.34}$$

$$(2) \quad \Gamma = \mathbb{1},\, \gamma_5,\, \sigma^{\mu\nu} \qquad \Rightarrow \qquad \begin{array}{l} \mathsf{P}_{-\omega} \, \Gamma \, \mathsf{P}_\omega = 0, \\ \bar{\psi}_{\omega}\, \Gamma \, \psi_\omega = 0 \end{array} \qquad \Rightarrow \quad \bar{\psi}\, \Gamma\, \psi = \sum_\omega \bar{\psi}_{-\omega} \, \Gamma\, \psi_\omega\,. \tag{2.35}$$

The elements in the first row lead to chirally symmetric terms in the Lagrangian:

$$\sum_\omega \bar{\psi}'_\omega \, \Gamma\, \psi'_\omega = \sum_\omega \bar{\psi}_\omega \, U_\omega^\dagger \, \Gamma\, U_\omega \, \psi_\omega = \sum_\omega \bar{\psi}_\omega \, \Gamma \, \psi_\omega\,; \tag{2.36}$$

but those in the second do not because $U_{-\omega}^\dagger \, U_\omega \neq \mathbb{1}$. For example, the kinetic term $\bar{\psi}\, i\rlap{/}\partial\, \psi$ is chirally invariant whereas a quark mass term $\sim m\,\mathbb{1}$ breaks chiral symmetry explicitly since it mixes right- and left-handed components.

Away from the chiral limit we can write the general Lagrangian (2.6) as

$$\mathcal{L} = \sum_\omega \left(\bar{\psi}_\omega \, i\rlap{/}\partial\, \psi_\omega - \bar{\psi}_{-\omega}\, \mathsf{M}\, \psi_\omega \right). \tag{2.37}$$

From the global $SU(N_f) \times SU(N_f)$ transformations we can define $2 \times (N_f^2 - 1)$ currents, which are however not conserved because the mass term mixes right- and left-handed quarks. Inserting the Dirac equations for ψ_ω and $\bar{\psi}_\omega$, their divergences are obtained as

$$j^\mu_{a,\omega} = \bar{\psi}_\omega \gamma^\mu \, \mathsf{t}_a \, \psi_\omega\,, \qquad \partial_\mu \, j^\mu_{a,\omega} = i \left(\bar{\psi}_{-\omega}\, \mathsf{M}\, \mathsf{t}_a \, \psi_\omega - \bar{\psi}_\omega \, \mathsf{t}_a \, \mathsf{M}\, \psi_{-\omega} \right). \tag{2.38}$$

In the chiral limit this leads to $2 \times (N_f^2 - 1)$ conserved chiral currents and charges:

$$j^\mu_{a,\omega} = \overline{\psi}_\omega \gamma^\mu \, \mathsf{t}_a \, \psi_\omega \,, \qquad \partial_\mu \, j^\mu_{a,\omega} = 0 \,, \qquad Q_{a,\omega} = \int d^3x \, \psi^\dagger_\omega \, \mathsf{t}_a \, \psi_\omega \,. \tag{2.39}$$

The vector and axialvector currents from Eqs. (2.19) and (2.25) and corresponding charges are linear combinations of the left- and right-handed currents and charges:

$$\begin{aligned} V^\mu_a &= j^\mu_{a,+} + j^\mu_{a,-} \,, & \qquad Q^V_a &= Q_{a+} + Q_{a-} \,, \\ A^\mu_a &= j^\mu_{a,+} - j^\mu_{a,-} \,, & \qquad Q^A_a &= Q_{a+} - Q_{a-} \,. \end{aligned} \tag{2.40}$$

Charge and current algebra. We will often need equal-time commutation relations for the currents in (2.12)–(2.13). Using the identity

$$[AB, CD] = A \{B, C\} \, D - C \{A, D\} \, B - \frac{\{A, C\} \, [B, D] + [A, C] \, \{B, D\}}{2} \tag{2.41}$$

together with the anticommutation relations (1.57) for the quark fields, and the (anti-) commutation relations (A.2) and (A.7) for the $SU(N)$ generators, it is easy to derive the generic relations

$$\begin{aligned} \Big[j^\Gamma_a(x), j^{\Gamma'}_b(y) \Big]_{x^0=y^0} &= \left[i f_{abc} \, j^{\Gamma_+}_c(x) + d_{abc} \, j^{\Gamma_-}_c(x) + \frac{\delta_{ab}}{N} \, j^{\Gamma_-}(x) \right] \delta^3(\boldsymbol{x} - \boldsymbol{y}), \\ \Big[j^\Gamma_a(x), j^{\Gamma'}(y) \Big]_{x^0=y^0} &= 2 j^{\Gamma_-}_a(x) \, \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \end{aligned} \tag{2.42}$$

where $\Gamma_\pm := \frac{1}{2} \, (\Gamma \gamma^0 \, \Gamma' \pm \Gamma' \gamma^0 \, \Gamma)$. They are valid independently of whether the currents are conserved or not.[2] For example, with Γ, $\Gamma' \in \{\gamma^0, \gamma^0 \gamma_5\}$ we arrive at

$$\begin{aligned} [V^0_a(x), V^0_b(y)]_{x^0=y^0} &= i f_{abc} \, V^0_c(x) \, \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \\ [V^0_a(x), A^0_b(y)]_{x^0=y^0} &= i f_{abc} \, A^0_c(x) \, \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \\ [A^0_a(x), A^0_b(y)]_{x^0=y^0} &= i f_{abc} \, V^0_c(x) \, \delta^3(\boldsymbol{x} - \boldsymbol{y}) \,. \end{aligned} \tag{2.43}$$

Hence, V^0_a and A^0_a form a closed algebra since they obey equal-time commutation relations with the structure constants of the Lie algebra, and the Dirac $\delta-$functions additionally ensure that all commutators vanish for $x \neq y$. This is the so-called *local current algebra*. If we further integrate over $\int d^3x$ and $\int d^3y$, we obtain the corresponding charge algebra on the Hilbert space:

$$[Q^V_a, Q^V_b] = [Q^A_a, Q^A_b] = i f_{abc} \, Q^V_c \,, \qquad [Q^V_a, Q^A_b] = i f_{abc} \, Q^A_c \,. \tag{2.44}$$

Actually, since the $SU(N_f)_A$ symmetry is spontaneously broken in the chiral limit, the axial charges are not rigorously defined; it is then more practical to work directly with the time components of the currents. Using Eq. (2.42) together with $\Gamma = \gamma^0 \gamma_5$ and $\Gamma' = i\gamma_5$, we obtain $\Gamma_+ = 0$ and $\Gamma_- = -i$ and therefore

$$\big[Q^A_a, P_b(x) \big] = -i \left[d_{abc} \, S_c(x) + \frac{\delta_{ab}}{N} \, S(x) \right], \tag{2.45}$$

[2] We ignore possible ambiguities due to *Schwinger terms*, see for example Itzykson-Zuber (p.530) and Pokorski (p.348) for discussions.

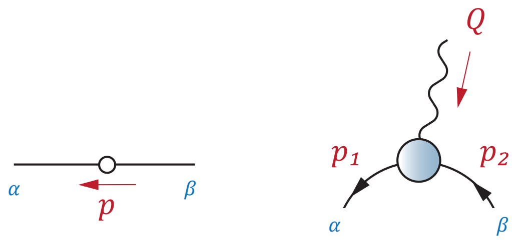

FIGURE 2.1: Quark propagator and three-point function in Eqs. (2.48) and (2.49).

where $S(x) = \overline{\psi}(x)\,\psi(x)$ is the scalar density defined via Eqs. (2.12)–(2.13). Its vacuum expectation value is the scalar quark condensate which will turn out to be nonvanishing due to spontaneous chiral symmetry breaking, and we will use this relation later for proving Goldstone's theorem and deriving the Gell-Mann-Oakes-Renner relation.

Using the relation $[AB,C] = A\{B,C\} - \{A,C\}B$, we can similarly obtain the commutation relations of the currents with the quark fields:

$$\begin{aligned} \big[\,j_a^\Gamma(x),\psi(y)\big]_{x^0=y^0} &= -\big(\mathsf{t}_a\,\gamma^0\,\Gamma\,\psi(x)\big)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \\ \big[\,j_a^\Gamma(x),\overline{\psi}(y)\big]_{x^0=y^0} &= \big(\overline{\psi}(x)\,\Gamma\,\gamma^0\,\mathsf{t}_a\big)\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \end{aligned} \tag{2.46}$$

If we integrate over $\int d^3x$, we get the commutation relations of the charges with the fields, for example for the vector currents ($\Gamma = \gamma^\mu$):

$$\big[\,Q_a^V(x_0),\psi(x)\big] = -\mathsf{t}_a\,\psi(x)\,, \qquad \big[\,Q_a^V(x_0),\overline{\psi}(x)\big] = \psi(x)\,\mathsf{t}_a\,. \tag{2.47}$$

They tell us once again that the charges are the infinitesimal generators of the symmetry group when acting on the Hilbert space.

Ward-Takahashi identities. At the level of Green functions, current conservation is manifest via Ward-Takahashi identities (WTIs) which are symmetry relations between the $n-$point and $(n+1)-$point functions. Take for example the quark propagator,

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|\,\mathsf{T}\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)\,|0\rangle. \tag{2.48}$$

How the quark couples to a vector, axialvector, scalar or pseudoscalar current (e.g. photons, $Z-$bosons, pions, ...) is encoded in the three-point function

$$(G_\Gamma)_{a,\alpha\beta}(x,x_1,x_2) := \langle 0|\mathsf{T}\,j_a^\Gamma(x)\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)|0\rangle\,, \tag{2.49}$$

with $j^\Gamma \in \{V^\mu, A^\mu, S, P\}$, see Fig. 2.1. In the vector and axialvector case, these two quantities are related by a Ward-Takahashi identity which we now want to establish. Consider two generic field operators $j^\mu(x)$, $\varphi(y)$ at different spacetime points. The divergence of their time-ordered product with respect to x is:

$$\begin{aligned} \partial_\mu^x\left(\mathsf{T}\,j^\mu(x)\,\varphi(y)\right) &= \partial_\mu^x\left(\Theta(x^0-y^0)\,j^\mu(x)\,\varphi(y) + \Theta(y^0-x^0)\,\varphi(y)\,j^\mu(x)\right) \\ &= \mathsf{T}\left(\partial_\mu\,j^\mu(x)\right)\varphi(y) + \delta(x^0-y^0)\left[j^0(x),\varphi(y)\right]. \end{aligned} \tag{2.50}$$

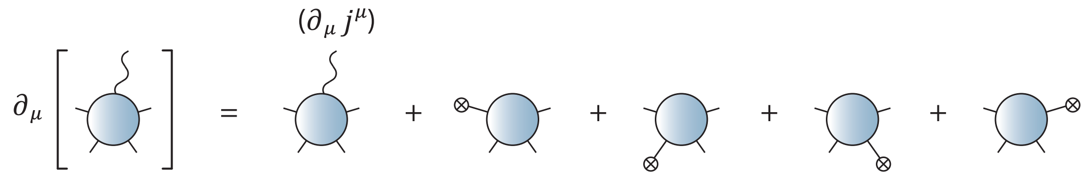

FIGURE 2.2: Generic form of a Ward-Takahashi identity from Eq. (2.52).

The first term comes from the derivative of $j^\mu(x)$ (simply resum the time orderings) and the second one results from differentiating the step functions:

$$\partial_\mu^x\,\Theta(x^0-y^0) = -\partial_\mu^x\,\Theta(y^0-x^0) = \delta(x^0-y^0)\,\delta_{0\mu}\,. \tag{2.51}$$

Eq. (2.50) is quite general and retains its structure for products of n different fields (which can also be fermionic). In the general case one has to write down all possible time orderings; the time-ordering of $n+1$ distinct space-time points leads to $(n+1)!$ terms, each of which includes products of n step functions. If fermion fields are involved, the individual time-ordered terms will pick up minus signs arising from the anticommutativity. In either case, the final result is

$$\begin{aligned}\partial_\mu^x\left(\mathsf{T}\,j^\mu(x)\,\varphi_1(x_1)\dots\varphi_n(x_n)\right) &= \mathsf{T}\left(\partial_\mu\,j^\mu(x)\right)\varphi_1(x_1)\dots\varphi_n(x_n)\\ &+\sum_{k=1}^n \delta(x^0-x_k^0)\,\mathsf{T}\,\varphi_1(x_1)\dots\left[j^0(x),\varphi_k(x_k)\right]\dots\varphi_n(x_n)\,.\end{aligned} \tag{2.52}$$

If we take its vacuum expectation value, it relates the $(n+1)-$point function, where one leg corresponds to the external current, to the $n-$point functions since the commutators in the second row will be proportional to the fields, cf. (2.46). This is the generic form of a Ward-Takahashi identity, illustrated in Fig. 2.2. Current conservation (or its absence) enters only in the first term on the right-hand side which vanishes if the current is conserved.

Let's work out this formula for the three-point function in Eq. (2.49):

$$\begin{aligned}\partial_\mu^x\,G_\Gamma^\mu(x,x_1,x_2) = &\langle 0|\,\mathsf{T}\left(\partial_\mu\,j^\mu(x)\right)\,\psi(x_1)\,\overline{\psi}(x_2)\,|0\rangle\\ &+\delta(x^0-x_1^0)\langle 0|\,\mathsf{T}\left[j^0(x),\psi(x_1)\right]\overline{\psi}(x_2)\,|0\rangle\\ &+\delta(x^0-x_2^0)\langle 0|\,\mathsf{T}\psi(x_1)\left[j^0(x),\overline{\psi}(x_2)\right]\,|0\rangle\,.\end{aligned} \tag{2.53}$$

For each type of current we insert the respective result from Eq. (2.46). If we use vector current conservation (for equal quark masses) and the PCAC relation, we obtain the vector and axialvector WTIs:

$$\partial_\mu^x\,G_V^\mu(x,x_1,x_2) = -\delta^4(x-x_1)\,\mathsf{t}_a\,S(x_1,x_2)+\delta^4(x-x_2)\,S(x_1,x_2)\,\mathsf{t}_a\,, \tag{2.54}$$

$$\begin{aligned}\partial_\mu^x\,G_A^\mu(x,x_1,x_2) = &\,2m\,G_P(x,x_1,x_2)\\ &-\delta^4(x-x_1)\,\gamma_5\,\mathsf{t}_a\,S(x_1,x_2)-\delta^4(x-x_2)\,S(x_1,x_2)\,\mathsf{t}_a\,\gamma_5\,,\end{aligned} \tag{2.55}$$

where the generator $\mathtt{t}_a$ acts on the flavor index of ψ in the first term and $\bar{\psi}$ in the second. In momentum space, the derivative with respect to x becomes a contraction with the external momentum $Q = p_1 - p_2$:

$$\begin{aligned} iQ^\mu G_V^\mu(p_1,p_2) &= -S(p_1) + S(p_2)\,, \\ iQ^\mu G_A^\mu(p_1,p_2) &= 2m\, G_P(p_1,p_2) - \gamma_5\, S(p_1) - S(p_2)\,\gamma_5\,, \end{aligned} \tag{2.56}$$

where we suppressed the generators for simplicity. The first equation can be solved to obtain the most general vertex that is compatible with vector current conservation (the so-called *Ball-Chiu vertex*), apart from further transverse terms with respect to the external momentum.[3] The axialvector WTI relates the longitudinal part of the axialvector vertex with the pseudoscalar vertex and the quark propagator. Here we have considered only the flavor-octet current A_a^μ; in the flavor-singlet channel we would have an additional term from the anomaly.

WTIs from the generating functional. In principle, WTIs can also be derived from the generating functional via Eq. (1.74). The prototype is the local $U(1)$ gauge invariance in QED. The QED path integral has the same form as in QCD,

$$Z[J,\eta,\bar{\eta}\,] = \int \mathcal{D}[A,\psi,\bar{\psi}\,]\; e^{i\left(S[A,\psi,\bar{\psi}\,]+S_{\text{GF}}[A]+S_{\text{C}}\right)}, \tag{2.57}$$

with the exception that there are no ghosts (because the Faddeev-Popov determinant is independent of the photon field A^μ and can be pulled out of the path integral), and instead of the non-Abelian $SU(3)_C$ symmetry we have only a phase $U(x) = e^{i\varepsilon(x)}$. A gauge transformation is just a relabeling of fields under the integral (if we keep the sources fixed) and thereby leaves the generating functional invariant. Since the QED action is gauge invariant, and assuming that the integral measure remains invariant as well, this will only affect the gauge-fixing and source terms:

$$S_{\text{GF}} + S_{\text{C}} = \int d^4x \left[\frac{1}{2\xi}\, A^\mu \partial_\mu \partial_\nu A^\nu - J_\mu A^\mu - \bar{\psi}\,\eta - \bar{\eta}\,\psi\right]. \tag{2.58}$$

Hence, if we insert $\delta\psi = i\varepsilon\psi$, $\delta\bar{\psi} = -i\varepsilon\bar{\psi}$, $\delta A^\mu = \frac{1}{g}\,\partial^\mu\varepsilon$ and perform partial integrations to factor out $\varepsilon(x)$, we obtain

$$\langle\, \delta S_{\text{GF}} + \delta S_{\text{C}}\,\rangle_J = \int d^4x\, \varepsilon(x) \left\langle \frac{1}{g}\,\partial_\mu \left(J^\mu - \frac{1}{\xi}\,\Box A^\mu\right) + i\left(\bar{\psi}\,\eta - \bar{\eta}\,\psi\right) \right\rangle_J = 0\,. \tag{2.59}$$

Since $\varepsilon(x)$ is arbitrary, the integrand must vanish as well, so that we arrive at

$$\partial_\mu \left(J^\mu - \frac{1}{\xi}\,\Box\,\langle A^\mu\rangle_J\right) + ig\left(\langle\bar{\psi}\rangle_J\,\eta - \bar{\eta}\,\langle\psi\rangle_J\right) = 0\,. \tag{2.60}$$

We can now use Eq. (1.46) to replace $\langle A^\mu\rangle_J$, $\langle\psi\rangle_J$ and $\langle\bar{\psi}\rangle_J$ by the field expectation values (we omit the tilde), and Eq. (1.42) to replace the currents by the derivative of the effective action with respect to the fields: $J = \delta\Gamma/\delta A$, $\eta = \delta\Gamma/\delta\bar{\psi}$, $\bar{\eta} = -\delta\Gamma/\delta\psi$.

[3]Note that all quantities in these equations are *connected* Green functions; one can get their 1PI analogues simply by multiplying both equations with inverse quark propagators from the left and right.

Further field derivatives yield the Ward-Takahashi identities for the 1PI Green functions. For example, after applying two derivatives with respect to ψ and $\bar{\psi}$ one arrives at the WTI for the quark-photon vertex which is similar to that in Eq. (2.54). (It must be, since the photon is also a vector field).

Background fields. Now what if we are instead interested in *global* flavor symmetries? Let's check first the QED case with a global $U(1)$ symmetry instead of a local one. In that case, $\delta A^\mu = 0$, and since ε is a constant, we can no longer eliminate the integral in Eq. (2.59) but get instead:

$$\langle\, \delta S_{\text{GF}} + \delta S_{\text{C}}\,\rangle_J = i\varepsilon \int d^4x\, \langle \bar{\psi}\,\eta - \bar{\eta}\,\psi\rangle_J = 0\,. \tag{2.61}$$

This equation is correct but not very useful. In the context of Eq. (2.54) it only tells us that the integrated equation vanishes – or in momentum space, that the difference of propagators on the right-hand side vanishes if their momenta are equal ($Q^\mu = 0$).

We can cure the problem by tricking the path integral into believing that it deals with a *local* symmetry instead of a global one. Suppose we start from the free quark Lagrangian in Eq. (2.6). Let's omit the quark-gluon vertex because it is not relevant for the discussion:

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi\,, \qquad Z[\,\eta,\bar{\eta}\,] = \int \mathcal{D}[\,\psi,\bar{\psi}\,]\; e^{i\left(S[\,\psi,\,\bar{\psi}\,]+S_{\text{C}}\right)}. \tag{2.62}$$

The action $S[\psi,\,\bar{\psi}\,]$ is invariant under the global $SU(N_f)_V \times U(1)_V$ symmetry; we consider $U(1)_V$ for simplicity. Its flavor-singlet current $V^\mu = \bar{\psi}\,\gamma^\mu\,\psi$ is conserved. The idea is now to add source terms to the action and define appropriate gauge transformations for the source fields, so that the *total* action including all sources becomes *locally* gauge invariant with respect to $U(1)_V$. This means we need a covariant derivative; from Eq. (2.62) we only need to add a term $\bar{\psi}\,B\!\!\!/\,\psi = V\cdot B$ to establish local $U(1)$ gauge invariance:

$$Z[\,B,\eta,\bar{\eta}\,] = \int \mathcal{D}[\,\psi,\bar{\psi}\,]\; e^{i\left(S[\,\psi,\,\bar{\psi}\,]+V\cdot B+S_{\text{C}}\right)}. \tag{2.63}$$

Hence, B plays the role of the gauge field, but it is a 'background' field since it doesn't appear in the path integral measure: it doesn't change the content of the quantum field theory. From Eq. (1.18) we have $\delta B_\mu = \partial_\mu\varepsilon$ because we are dealing with an Abelian gauge symmetry (we set the irrelevant new coupling to 1). We can make the source term gauge invariant in itself by demanding that $\delta\eta = i\varepsilon\,\eta$ and $\delta\bar{\eta} = -i\varepsilon\,\bar{\eta}$.

Now start from $Z[\,B',\eta',\bar{\eta}'\,]$, relabel the fields in the path integral ψ, $\bar{\psi} \to \psi'$, $\bar{\psi}'$, and perform a gauge transformation back to unprimed quantities. The total action is invariant and the path integral measure as well, so that also the partition function is invariant under a change of B, η, $\bar{\eta}$. This leads to the condition

$$\int d^4x\,\langle V\cdot\delta B - \bar{\psi}\,\delta\eta - \delta\bar{\eta}\,\psi\rangle_J = \int d^4x\,\big[\langle V^\mu\rangle_J\,\partial_\mu\varepsilon - i\varepsilon\,\big(\langle\bar{\psi}\rangle_J\,\eta - \bar{\eta}\,\langle\psi\rangle_J\big)\big] = 0\,. \tag{2.64}$$

In order to arrive at Eq. (2.54) including connected Green functions, replace

$$\langle V^\mu\rangle_J = -\frac{\delta W}{\delta B_\mu}\,, \qquad \langle\psi\rangle_J = -\frac{\delta W}{\delta\bar{\eta}}\,, \qquad \langle\bar{\psi}\rangle_J = \frac{\delta W}{\delta\eta}\,, \tag{2.65}$$

and perform a partial integration. Since $\varepsilon(x)$ is again arbitrary one can remove the integral, and the resulting master WTI becomes

$$\partial_\mu \frac{\delta W}{\delta B_\mu} = i\left(\frac{\delta W}{\delta\eta}\,\eta + \bar\eta\,\frac{\delta W}{\delta\bar\eta}\right). \tag{2.66}$$

It has the same form as in our first attempt (2.61) except that we have now a new Green function $\delta W/\delta B$ that incorporates the current. The vector WTI (2.54) follows from applying two further derivatives with respect to η and $\bar\eta$ and setting the sources to zero.

Renormalization of currents. So far we have only dealt with bare currents that we derived from the bare Lagrangian (2.6). However, if we included renormalization constants for the vector and axialvector currents, the current-algebra relations (2.43) would fix both of them to $Z^2 = Z = 1$. Hence, these currents stay unrenormalized, which entails

$$\begin{aligned} V^\mu_\mathrm{B} &= (\bar\psi\,\gamma^\mu\,\psi)_\mathrm{B} = Z_2\,(\bar\psi\,\gamma^\mu\,\psi)_\mathrm{R} = V^\mu_\mathrm{R}\,, \\ A^\mu_\mathrm{B} &= (\bar\psi\,\gamma^\mu\gamma_5\,\psi)_\mathrm{B} = Z_2\,(\bar\psi\,\gamma^\mu\gamma_5\,\psi)_\mathrm{R} = A^\mu_\mathrm{R}\,. \end{aligned} \tag{2.67}$$

On the other hand, those relations do not give us closed equations for the scalar and pseudoscalar densities. In that case we can exploit the fact that their divergences are proportional to the quark masses, e.g., from the PCAC relation:

$$\partial_\mu A^\mu_\mathrm{B} = (2mP)_\mathrm{B} \stackrel{!}{=} (2mP)_\mathrm{R} = \partial_\mu A^\mu_\mathrm{R} \quad\Rightarrow\quad P_\mathrm{B} = \frac{1}{Z_m}\,P_\mathrm{R}\,, \tag{2.68}$$

and consequently

$$P_\mathrm{B} = (\bar\psi\gamma_5\psi)_\mathrm{B} = Z_2\,(\bar\psi\gamma_5\psi)_\mathrm{R} = \frac{1}{Z_m}\,P_\mathrm{R}\,. \tag{2.69}$$

The same result follows for the scalar density. In summary, the renormalized currents are (we drop the label 'R'):

$$V^\mu = Z_2\,\bar\psi\,\gamma^\mu\psi\,, \quad A^\mu = Z_2\,\bar\psi\,\gamma^\mu\gamma_5\,\psi\,, \quad P = Z_2 Z_m\,\bar\psi\gamma_5\psi\,, \quad S = Z_2 Z_m\,\bar\psi\psi\,. \tag{2.70}$$