2.2 Hadrons, poles and decay constants

We have mentioned the implications of various symmetry relations for hadrons, but we have not yet developed the tools to actually extract hadron properties from QCD. In principle, hadrons are contained in the state space of QCD. A self-adjoint Hamiltonian has a complete set of orthogonal eigenstates which we will call $|\lambda\rangle$; they carry momenta p plus further quantum numbers that reflect the symmetries of QCD (angular momentum, parity, flavor, etc.). Their completeness relation is

$$\mathbb{1} = \sum_\lambda \frac{1}{(2\pi)^3} \int d^4p\, \theta(p^0)\, \delta(p^2 - m_\lambda^2)\, |\lambda\rangle\langle\lambda| = \sum_\lambda \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_{\boldsymbol{p}}}\, |\lambda\rangle\langle\lambda|\,, \tag{2.71}$$

where the Lorentz-invariant integral weight implements the condition that each hadron is on its mass shell ($p^2 = m_\lambda^2$, or $E_{\boldsymbol{p}}^2 = \boldsymbol{p}^2 + m_\lambda^2$). You might understandably feel a bit uncomfortable with all this: in principle, the state space can contain (unphysical) colored states, colorless 'one-particle' bound states like mesons and baryons, but also glueballs, multiquark and multi-hadron states – also the C^{14} nucleus should be somewhere buried in the QCD state space. We will only be interested in $q\bar{q}$ and qqq color singlets, but whenever you encounter a sum over λ, keep in mind that the actual Fock space of QCD is *enormous*.

Hadrons generate poles. A useful way to extract hadron properties, which is also closely related to the experimental situation, is based on the fact that hadrons produce poles in QCD's Green functions, and hence in scattering amplitudes and cross sections. The starting point is the Källén-Lehmann spectral representation which is usually derived for the *propagator* of a theory. Inserting the completeness relation (2.71) between the two field operators that appear in the propagator's time-ordered vacuum expectation value yields a single-particle pole at $p^2 = m_\lambda^2$, and in principle also a multi-particle continuum with branch cuts that start at $p^2 = 4m_\lambda^2$ and extend to infinity. This property will, however, not hold in QCD because such states would carry color. Since quarks transform under the fundamental triplet representation of $SU(3)_C$, a single quark field operator cannot create colorless states, and one has to make sure somehow that those are indeed absent from the physical state space. In fact, the absence of a Källén-Lehmann representation can be used as a criterion for confinement: the elementary quark and gluon propagators should not have timelike particle poles.

On the other hand, bound states are color singlets and can appear as poles in higher $n-$point functions, which allows us to derive a spectral representation for those. Take for example the quark four-point function

$$G_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = \langle 0|\mathsf{T}\, \psi_\alpha(x_1)\, \overline{\psi}_\beta(x_2)\, \psi_\gamma(x_3)\, \overline{\psi}_\delta(x_4)|0\rangle\,. \tag{2.72}$$

Inserting a complete set of states will produce bound-state poles because a composite operator $\psi\overline{\psi}$ can produce color singlet quantum numbers ($3 \otimes \bar{3} = 1 \oplus 8$). Instead of working with the four-point function directly, we can simplify the problem by setting $x_1 = x_2$ and $x_3 = x_4$ and contracting the resulting quark pairs with Dirac and flavor matrices $\mathsf{t}_a\, \Gamma_{\beta\alpha}\, \Gamma'_{\delta\gamma}\, \mathsf{t}_b$ from Eq. (2.12). Then we obtain current correlators of the form

$$\langle 0|\mathsf{T} P_a(x) P_b(y)|0\rangle\,, \quad \langle 0|\mathsf{T}\, V_a^\mu(x)\, V_b^\nu(y)|0\rangle\,, \quad \langle 0|\mathsf{T}\, A_a^\mu(x)\, A_b^\nu(y)|0\rangle\,, \quad \text{etc.} \tag{2.73}$$

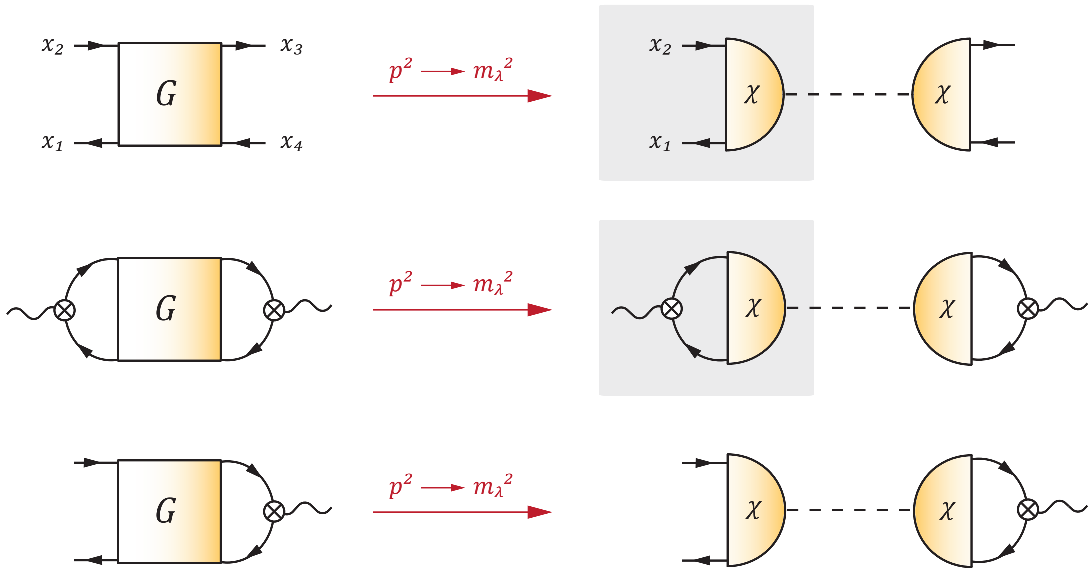

FIGURE 2.3: Quark four-point function (2.72) and its various Dirac-flavor contractions, and their separability at a given meson pole from the spectral representation. Second row: current correlators in (2.73), third row: vertices from (2.49). The symbol $\otimes$ represents a Dirac-flavor matrix $\Gamma\,\mathsf{t}_a$. The highlighted quantities are the bound-state wave function (2.74) and the decay constants from Eq. (2.78). The dashed lines are Feynman propagators.

These are again two-point functions and can be viewed as effective meson propagators since they contain the composite fields P_a, V_a^μ, A_a^μ, etc. This is also a convenient way to filter the overwhelming information from the state space, because poles will only emerge from those states which coincide with the quantum numbers of the currents. Another advantage is that the Green functions in (2.73) are already gauge-invariant (in contrast to the four-point function) since they contain gauge-invariant, local products of quark fields. This is the strategy that is usually pursued in lattice calculations, since the properties of such correlators at the pole (in coordinate space: the large Euclidean time behavior) can be calculated directly from the QCD partition function.

Bound-state wave function. Let's work this out in more detail. So far we have only looked at vacuum-to-vacuum transition matrix elements of time-ordered operators, i.e., the Green functions of the theory. Now we introduce transition elements between the vacuum and a hadron with momentum p, which define a hadron's bound-state (or Bethe-Salpeter) wave function:

$$\chi^a_{\alpha\beta}(x_1, x_2, p) := \langle 0|\, \mathsf{T}\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\, |\lambda_a\rangle\,. \tag{2.74}$$

Since this quantity contains a quark and antiquark field it corresponds to a meson; we could also write down the analogue for baryons with three quark fields. The hadron's total momentum is onshell ($p^2 = m_\lambda^2$) and all other quantum numbers are fixed.

Translation invariance entails that a usual vacuum-to-vacuum transition amplitude can only depend on the relative coordinate $z := x_1 - x_2$ but not on the total position $x := (x_1 + x_2)/2$. For a vacuum-to-hadron amplitude as in Eq. (2.74), it means that the dependence on x can only enter through a phase (cf. Eq. (B.71) in the Appendix):

$$\chi^a_{\alpha\beta}(x_1, x_2, p) = \langle 0|\, \mathsf{T}\psi_\alpha\big(\tfrac{z}{2}\big)\,\overline{\psi}_\beta\big(-\tfrac{z}{2}\big)\,|\lambda_a\rangle\, e^{-ip\cdot x} = \chi^a_{\alpha\beta}(z,p)\, e^{-ip\cdot x}\,. \tag{2.75}$$

To see this, use $x_1 = x + \frac{z}{2}$, $x_2 = x - \frac{z}{2}$ and insert the behavior of field operators and one-particle states under a Poincaré transformation $U(\Lambda, a)$ which is a pure translation:

$$U(1,a)\,\psi_\alpha(x)\,U(1,a)^{-1} = \psi_\alpha(x+a)\,, \quad U(1,a)\,|\lambda(p)\rangle = e^{ip\cdot a}\,|\lambda(p)\rangle\,, \tag{2.76}$$

and use translation invariance of the vacuum: $U(1,a)\,|0\rangle = |0\rangle$.

Decay constants. We can extract gauge-invariant quantities from the bound-state wave function by setting $z = 0$ (in momentum space, this means integration over the relative momentum) and contracting it with some Dirac-flavor structure $\Gamma\,\mathsf{t}_a$. This yields the vacuum-to-hadron transition element of the corresponding current:

$$-\,\Gamma_{\beta\alpha}\,\mathsf{t}_a\,\chi^b_{\alpha\beta}(x,x,p) = \langle 0|\, j^\Gamma_a(x)\,|\lambda_b\rangle = \langle 0|\, j^\Gamma_a(0)\,|\lambda_b\rangle\, e^{-ip\cdot x}\,. \tag{2.77}$$

Take for example $\Gamma = \gamma^\mu\gamma_5$ and $i\gamma_5$ which produce axialvector and pseudoscalar currents. This restricts the possibilities for $|\lambda_a\rangle$ to pseudoscalar and axialvector mesons; for the moment we will consider pseudoscalars only. Since we also take the flavor trace of two generators, the only possible structure in flavor space is $\sim \delta_{ab}$, cf. (A.6):

$$\langle 0|\, A^\mu_a(x)\,|\lambda_b\rangle = \delta_{ab}\, ip^\mu f_\lambda\, e^{-ip\cdot x}, \qquad \langle 0|\, P_a(x)\,|\lambda_b\rangle = \delta_{ab}\, r_\lambda\, e^{-ip\cdot x}. \tag{2.78}$$

The first quantity encodes the transition from a pseudoscalar meson to an axialvector current. Since the pion ($\lambda = \pi$) decays weakly into leptons ($\pi^+ \to W^+ \to \mu^+ + \nu_\mu$), it defines the pion's electroweak decay constant f_π. Its generic structure arises from the fact that the quantity is a Lorentz vector whose only possible tensor structure is p^μ, and the pion decay constant can in principle depend on the Lorentz-invariant p^2 but $p^2 = m_\pi^2$ is fixed. The pseudoscalar analogue r_λ is not directly measurable but will be useful in the following. If we now apply the PCAC relation (2.26) for equal quark masses, we immediately obtain

$$f_\lambda\, m_\lambda^2 = 2m\, r_\lambda\,, \tag{2.79}$$

which is valid for all flavor non-singlet pseudoscalar mesons. (In the singlet case, there would be an additional term from the anomaly.) For example, it relates the pion decay constant and pion mass with the pseudoscalar transition matrix element r_π. This already resembles the Gell-Mann-Oakes-Renner (GMOR) relation, but so far we know nothing about spontaneous chiral symmetry breaking! At this point, the equation only tells us that in the chiral limit ($m = 0$) either $f_\lambda = 0$ or $m_\lambda = 0$.[4]

[4] In Sec. 2.4 we will prove the Goldstone theorem and derive the GMOR relation. The essence of the proof is to show that the pion decay constant does *not* vanish in the chiral limit (as a consequence of spontaneous chiral symmetry breaking), and therefore $m_\pi = 0$.

With the same reasoning, Lorentz covariance and parity invariance also settle the general structure of a pseudoscalar-meson wave function in momentum space:

$$\chi^a(q,p) = i\gamma_5 \left(h_1 + h_2 \not{p} + h_3 \not{q}_\perp + h_4 [\not{q}, \not{p}]\right) \mathsf{t}_a. \tag{2.80}$$

Here $p = p_1 - p_2$ is the total momentum and $q = (p_1 + p_2)/2$ the relative momentum. The $h_i(q^2, q\cdot p, p^2)$ are Lorentz-invariant functions of all invariant momentum variables. $q_\perp$ is orthogonalized with respect to p so that all four tensor structures are orthogonal to each other when taking the Dirac trace. As a consequence, only h_1 and h_2 survive when we integrate over the relative momentum q. In order to get (2.78), we would integrate over q (in coordinate space, we set $z = 0$) and take the trace with $\gamma^\mu\gamma_5$ and $i\gamma_5$, which projects out h_2 and h_1, respectively. Hence, even if the wave function itself is gauge and renormalization-point dependent, the integrated dressing functions $r_\lambda \sim \int d^4q\, h_1$ and $f_\lambda \sim \int d^4q\, h_2$ carry gauge-invariant and renormalization-point independent information.[5]

Current correlators. Consider now the pseudoscalar correlator in Eq. (2.73); we want to show that each bound state generates a pole in this Green function. Let's write the time orderings explicitly:

$$\begin{aligned}\langle 0|\mathsf{T}\, P_a(x)\, P_b(y)|0\rangle = \Theta(x^0 - y^0)\, \langle 0|\, P_a(x)\, P_b(y)\, |0\rangle \\ + \Theta(y^0 - x^0)\, \langle 0|\, P_b(y)\, P_a(x)\, |0\rangle\,.\end{aligned} \tag{2.81}$$

If we insert the completeness relation (2.71) and use (2.78), we obtain ($z = x - y$):

$$\begin{aligned}\langle 0|\mathsf{T} P_a(x) P_b(y)|0\rangle &= \sum_\lambda \left[\int \frac{d^3p}{2E_{\boldsymbol{p}}}\, \frac{\Theta(z^0)\, e^{-ipz} + \Theta(-z^0)\, e^{ipz}}{(2\pi)^3}\right] r_\lambda^2\, \delta_{ab} \\ &= \sum_\lambda D_F(z, m_\lambda)\, r_\lambda^2\, \delta_{ab} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipz} \sum_\lambda \frac{i r_\lambda^2\, \delta_{ab}}{p^2 - m_\lambda^2 + i\varepsilon}\end{aligned} \tag{2.82}$$

because the square bracket in the first line is just the definition of the Feynman propagator:

$$D_F(z, m_\lambda) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipz}\, \frac{i}{p^2 - m_\lambda^2 + i\varepsilon}\,. \tag{2.83}$$

Eq. (2.82) is the Källén-Lehmann representation for the pseudoscalar current correlator: it can be expressed by a sum over particle poles with masses m_λ and residues r_λ^2. The sum over λ retains only those states which overlap with the pseudoscalar density, i.e., the pseudoscalar mesons. In principle we should generalize formulas like these to a spectral density $\rho(m^2)$ in order to include the various branch cuts from the multiparticle continua:

$$\sum_\lambda D_F(z, m_\lambda)\, R_\lambda \to \int dm^2\, D_F(z,m) \Big[\underbrace{\sum_\lambda R_\lambda\, \delta(m^2 - m_\lambda^2) + (\dots)}_{=:\rho(m^2)}\Big]. \tag{2.84}$$

[5] In the case of r_λ, we have to multiply with the quark mass to make this statement exact, cf. (2.79).

One can repeat the derivation also for mixed correlators, for example:

$$
\begin{aligned}
\langle 0|\mathsf{T}\, A_a^\mu(x)\, P_b(y)|0\rangle &= \sum_\lambda \left[\int \frac{d^3p}{2E_{\boldsymbol{p}}}\, \frac{\Theta(z^0)\, e^{-ipz} - \Theta(-z^0)\, e^{ipz}}{(2\pi)^3}\right] ip^\mu f_\lambda\, r_\lambda\, \delta_{ab} \\
&= -\frac{\partial}{\partial z_\mu} \sum_\lambda D_F(z, m_\lambda)\, f_\lambda\, r_\lambda\, \delta_{ab} \\
&= -\int \frac{d^4p}{(2\pi)^4}\, e^{-ipz} \sum_\lambda \frac{p^\mu f_\lambda\, r_\lambda\, \delta_{ab}}{p^2 - m_\lambda^2 + i\varepsilon}\,.
\end{aligned} \tag{2.85}
$$

A timelike pole in momentum space corresponds to an exponential (Euclidean) time decay. This can be seen from the spatial integral over the Feynman propagator (2.83):

$$
\int d^3z\, D_F(z, m_\lambda) \xrightarrow{\text{Euclidean}} \int \frac{dp_0}{2\pi}\, \frac{e^{-ip_0 z_0}}{p_0^2 + m_\lambda^2} = \frac{e^{-m_\lambda |z_0|}}{2m_\lambda}\,. \tag{2.86}
$$

If we put this back in the spectral representation and take the sum over λ, the ground state with lowest mass will dominate the sum at large times. In this way hadron masses and other observables can be extracted in lattice QCD.

Hadron poles are everywhere! Current correlators are the Dirac-flavor traced versions of the quark four-point function from Eq. (2.72). While they contain only poles with the same quantum numbers as the involved currents, the four-point function contains *all* possible meson poles. In this case, working out the spectral representation is more tedious because one first has to write down all possible ($4! = 24$) time orderings to ensure the correct arrangements of the step functions that will eventually constitute the Feynman propagator. It turns out that these 24 step functions can be grouped into three categories which correspond to the s, t and u channels. If we abbreviate $\psi_\alpha(x_1) \equiv \psi_1$ and so on, and introduce the variables

$$
\begin{aligned}
z_s &= x_1 + x_2 - x_3 - x_4 \qquad (s \text{ channel})\,, \\
z_t &= x_1 + x_4 - x_3 - x_2 \qquad (t \text{ channel})\,, \\
z_u &= x_1 + x_3 - x_2 - x_4 \qquad (u \text{ channel})\,,
\end{aligned} \tag{2.87}
$$

then after some algebra[6] you can verify that $G_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4)$ can be written as

$$
\begin{aligned}
G_{1234} = \;&\Theta(z_s^0)\, \langle 0|\mathsf{T}(\psi_1\, \overline{\psi}_2)\, \mathsf{T}(\psi_3\, \overline{\psi}_4)|0\rangle + \Theta(-z_s^0)\, \langle 0|\mathsf{T}(\psi_3\, \overline{\psi}_4)\, \mathsf{T}(\psi_1\, \overline{\psi}_2)|0\rangle \\
&- \Theta(z_t^0)\, \langle 0|\mathsf{T}(\psi_1\, \overline{\psi}_4)\, \mathsf{T}(\psi_3\, \overline{\psi}_2)|0\rangle - \Theta(-z_t^0)\, \langle 0|\mathsf{T}(\psi_3\, \overline{\psi}_2)\, \mathsf{T}(\psi_1\, \overline{\psi}_4)|0\rangle \\
&- \Theta(z_u^0)\, \langle 0|\mathsf{T}(\psi_1\, \psi_3)\, \mathsf{T}(\overline{\psi}_2\, \overline{\psi}_4)|0\rangle - \Theta(-z_u^0)\, \langle 0|\mathsf{T}(\overline{\psi}_2\, \overline{\psi}_4)\, \mathsf{T}(\psi_1\, \psi_3)|0\rangle\,.
\end{aligned} \tag{2.88}
$$

Inserting the completeness relation in any of these channels yields then the spectral representation of the four-point function, where the pole residues are now the bound-state wave functions from Eq. (2.74). For example, the $s-$channel contribution becomes in momentum space:

$$
G_{\alpha\beta\gamma\delta}(q, q', p)\Big|_{s\,\text{channel}} = \sum_\lambda \frac{i\chi^a_{\alpha\beta}(q, p)\, \overline{\chi}^a_{\gamma\delta}(q', p)}{p^2 - m_\lambda^2 + i\varepsilon}\,, \tag{2.89}
$$

[6]Use the fact that $a > b > c > d$ is equivalent to $(a > b)$, $(a + b > c + d)$, $(c > d)$, and hence $\Theta(a-b)\,\Theta(b-c)\,\Theta(c-d) = \Theta(a+b-c-d)\,\Theta(a-b)\,\Theta(c-d)$.

where q and q' are the respective relative momenta. An analogous formula holds for the t channel. In the $u-$channel, one would obtain diquark poles which are forbidden because diquarks are not color singlets. Similar relations are then also valid for the quark six-point function (with baryon poles), for the eight-point function (tetraquark poles?), and in principle also for the 12-point function (deuteron) etc.

As another example, you can also contract only *one* $q\bar{q}$ pair in the four-point function, as shown in the bottom line of Fig. 2.3: this yields the three-point vertices defined earlier in Eq. (2.49). Consequently, meson poles will also show up in those Green functions. Depending on the flavor generator t_a of the current, the vector vertex G_V^μ will contain vector meson poles, the pseudoscalar vertex pseudoscalar mesons, and so on, whenever the square of the total momentum approaches $Q^2 \to m_\lambda^2$.

Finally, poles will also appear in the five-point function

$$\begin{aligned}(G_\Gamma)_{a,\alpha\beta\gamma\delta}(x,x_1,x_2,x_3,x_4) &:= \langle 0|\mathsf{T}\, j_a^\Gamma(x)\,\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)\,\psi_\gamma(x_3)\,\bar{\psi}_\delta(x_4)\,|0\rangle\,, \\ (G_\Gamma)_{a,\alpha\beta\gamma\delta}(q,q',p,p')\Big|_{s\,\text{channel}} &= \sum_{\lambda\lambda'} \frac{i\chi^b_{\alpha\beta}(q,p)}{p^2-m_\lambda^2+i\varepsilon}\,\langle\lambda_b|\,j_a^\Gamma(0)\,|\lambda_c'\rangle\,\frac{i\bar{\chi}^c_{\gamma\delta}(q',p')}{p'^2-m_{\lambda'}^2+i\varepsilon}\,.\end{aligned} \tag{2.90}$$

Its residue $\langle\lambda_b|\,j_a^\Gamma(0)\,|\lambda_c'\rangle$ defines a hadron's current matrix element, or the transition current matrix element between two different hadrons. Its decomposition in momentum space, similarly to (2.78) and (2.80), encodes the various measurable form factors of hadrons: electromagnetic, axial, pseudoscalar, scalar form factors, etc.