## 2.3 Hadron spectrum

We have studied the flavor structure of the QCD Lagrangian and its group-theoretical implications for hadron properties as well as for currents and for $n-$point functions. Now it's time for a reality check, because in principle the various symmetries of the Lagrangian should be reflected in the hadron spectrum:

- **SU(3) color gauge invariance:** Hadrons must be colorless, so they can only appear in the singlet representation of $SU(3)$. Singlets can be obtained by combining quarks and antiquarks to mesons, or three quarks to baryons:

$$\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\,,\qquad \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,, \tag{2.91}$$

  and they also appear in higher patterns of these combinations ($qq\overline{qq}$ tetraquarks, $qqqq\bar{q}$ pentaquarks, etc). Color singlets also arise from combining two (or more) gluons, which leads to the notion of glueballs:[7]

$$\mathbf{8}\otimes\mathbf{8}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27}\,. \tag{2.92}$$

  If we change the number of colors from three to $N_c$, the nature of a 'hadron' will change as well, cf. Table A.2:

$$N_c\otimes\overline{N}_c=\mathbf{1}\oplus\dots\,,\qquad \underbrace{N_c\otimes\dots\otimes N_c}_{N_c\text{ times}}=\mathbf{1}\oplus\dots\,, \tag{2.93}$$

  which means that mesons will still survive as $q\bar{q}$ states, but baryons will be bound states of $N_c$ quarks instead of three.

- **Flavor symmetries:** In our discussion of the currents in Section 2.1 we have seen that the various flavor symmetries can have implications for the hadron spectrum even if they are broken. The usual $SU(N_f)_V$ flavor symmetry allows us to classify hadrons in flavor multiplets where, in contrast to color, all combinations are allowed. In the three-flavor case mesons will form color singlets and octets, whereas baryons come in singlets, octets and decuplets (see Fig. 2.4):

$$\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\,,\qquad \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,. \tag{2.94}$$

  The states within these multiplets are labeled by the third isospin component $I_3$ and the hypercharge $Y$ (or equivalently, the strangeness), which are conserved quantum numbers even if the $SU(3)$ flavor symmetry is broken. In fact, the observation that hadrons appear in the $SU(3)$ octet, decuplet and singlet representations but *not* in the fundamental representation was the starting point for the development of the quark model.

[7] Consult Appendix A.2 for working out the product representations of $SU(N)$.

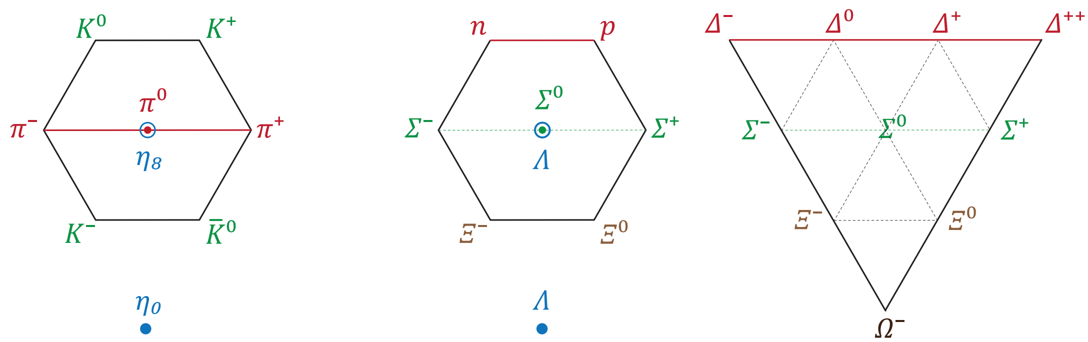


FIGURE 2.4: $SU(3)_F$ meson singlet and octet; baryon singlet, octet and decuplet.

- **Poincaré invariance:** The invariance of the QCD action under the Poincaré group gives us two quantum numbers to label the states, namely the eigenvalues of its Casimir operators: the total angular momentum ('spin') $J$ and the mass $M$ (see App. B). Together with parity invariance of the strong interaction, this allows us to arrange hadrons according to their $J^P$ quantum numbers. We find scalar ($0^+$), pseudoscalar ($0^-$), vector ($1^-$), axialvector ($1^+$), tensor ($2^+$) mesons and more, whereas the possible $J^P$ values for baryons are $1/2^\pm$, $3/2^\pm$, $5/2^\pm$, etc.

- **Charge-conjugation invariance:** Charge conjugation exchanges a particle with its antiparticle and therefore reverses $n_q$ for all flavors, the number of quarks minus antiquarks: $U_c\,|n_u, n_d, n_s, \dots\rangle = |-n_u, -n_d, -n_s, \dots\rangle$. Since $B$, $I_3$, $Y$ and $Q$ will then be reversed as well, only states for which all these additive quantum numbers vanish can be $C-$parity eigenstates. These are the neutral mesons, which are their own antiparticles and can be classified according to $J^{PC}$. Applying $U_c$ twice reverts the state back to its original one ($U_c^2 = 1$), so its possible eigenvalues are $C = \pm 1$. From the transformation properties of the quark fields,

$$U_c\,\psi\,U_c^{-1} = \eta_c\,(\bar{\psi}\,C)^T\,, \qquad U_c\,\bar{\psi}\,U_c^{-1} = \eta_c^\star\,(C\psi)^T \qquad C = i\gamma^2\gamma^0, \tag{2.95}$$

together with their anticommutativity, one can show that the Lagrangian is charge-conjugation invariant ($\eta_c$ is a phase factor). One can also work out the transformation behavior of the currents:

$$S \to S, \qquad P \to P, \qquad V^\mu \to -V^\mu, \qquad A^\mu \to A^\mu\,. \tag{2.96}$$

Mesons that are created by these currents carry therefore the quantum numbers $J^{PC} = 0^{++}$, $0^{-+}$, $1^{--}$ and $1^{++}$.

Experimentally, hadrons do indeed come in $J^{P(C)}$ multiplets. For each channel we find $SU(3)$ octets and singlets for mesons, and octets, decuplets and singlets for baryons. These multiplets appear as ground states but also as radial excitations. The current experimental status on meson and baryon masses is collected in Tables 2.2 and 2.4.

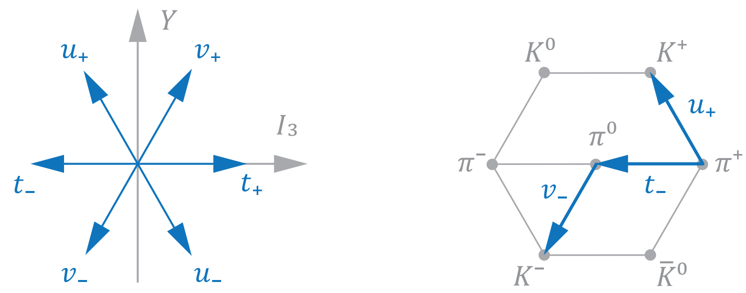


FIGURE 2.5: Construction of the octet with ladder operators.

### 2.3.1 Light mesons

**Flavor wave functions for mesons.** In principle the flavor wave functions for mesons can be read off already from the generators of the group $SU(N_f)$. Since the currents and charges define representations of their algebra on the state space, the flavor content of the generators will be inherited by the mesons that they create out of the vacuum. If we denote the $SU(3)$ flavor basis states by

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{aligned} \bar{u} &= (1,\, 0,\, 0), \\ \bar{d} &= (0,\, 1,\, 0), \\ \bar{s} &= (0,\, 0,\, 1), \end{aligned} \tag{2.97}$$

then the $\pi^+$ wave function is $u\bar{d} = u \otimes \bar{d} = \mathsf{t}_+$, etc. That is, we build the wave functions as tensors of mixed rank $(1,1)$ (cf. the discussion in App. A.3) that transform under the *reducible* representation $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$, and by orthogonalizing them in the end we single out the irreducible components.

The simplest construction principle is the one via ladder operators, see Fig. 2.5: in $SU(3)$ two of the generators, the isospin $\mathsf{t}_3$ and hypercharge $\mathsf{Y} = (2/\sqrt{3})\,\mathsf{t}_8$, label the states and the other six are ladder operators,

$$\mathsf{t}_\pm = \mathsf{t}_1 \pm i\mathsf{t}_2\,, \qquad \mathsf{u}_\pm = \mathsf{t}_6 \pm i\mathsf{t}_7\,, \qquad \mathsf{v}_\pm = \mathsf{t}_4 \pm i\mathsf{t}_5\,, \tag{2.98}$$

which lead away from the origin in the $(I_3, Y)$ plane. The Gell-Mann matrices act on the quark states via $\mathsf{t}_a$ from the left and on the antiquark states via $-\mathsf{t}_a^\star$ from the right. (The $-\mathsf{t}_a^\star$ satisfy the same commutation relations as the $\mathsf{t}_a$ and form the conjugate representation of the group.) For example, if we start from $\pi^+ = u\bar{d}$, we can successively generate the other meson states by acting with the ladder operators:

$$\begin{aligned} &\mathsf{t}_-\, u = d, \quad \bar{d}\,(-\mathsf{t}_-^\star) = -\bar{u} \quad \Rightarrow \quad \big(\mathsf{t}_- \otimes \mathbb{1} - \mathbb{1} \otimes \mathsf{t}_-^\star\big)\, u\bar{d} = d\bar{d} - u\bar{u} \quad \sim \pi^0\,, \\ &\mathsf{u}_+\, u = 0, \quad \bar{d}\,(-\mathsf{u}_+^\star) = -\bar{s} \quad \Rightarrow \quad \big(\mathsf{u}_+ \otimes \mathbb{1} - \mathbb{1} \otimes \mathsf{u}_+^\star\big)\, u\bar{d} = -u\bar{s} \quad \sim K^+\,, \end{aligned} \tag{2.99}$$

and so on. These states are subsequently normalized so that for example $\overline{(u\bar{d})}\,(u\bar{d}) = 1$. We can also check the eigenvalues of isospin and hypercharge (for mesons, hypercharge is identical to strangeness, cf. Eq. 2.22):

$$\begin{aligned} \mathsf{t}_3\, u = \tfrac{1}{2}\, u, \quad \bar{d}\,(-\mathsf{t}_3^\star) = +\tfrac{1}{2}\,\bar{d} \quad &\Rightarrow \quad (\mathsf{t}_3 \otimes \mathbb{1} - \mathbb{1} \otimes \mathsf{t}_3^\star)\, u\bar{d} = u\bar{d}\,, \\ \mathsf{Y}\, u = \tfrac{1}{3}\, u, \quad \bar{d}\,(-\mathsf{Y}^\star) = -\tfrac{1}{3}\,\bar{d} \quad &\Rightarrow \quad (\mathsf{Y} \otimes \mathbb{1} - \mathbb{1} \otimes \mathsf{Y}^\star)\, u\bar{d} = 0\,. \end{aligned} \tag{2.100}$$

| $0^-$ | $1^-$ | $I$ | $I_3$ | $S$ | | |
|---|---|---|---|---|---|---|
| $\pi^+$ | $\rho^+$ | 1 | 1 | 0 | $u\bar{d}$ | $\mathsf{t}_+$ |
| $\pi^0$ | $\rho^0$ | 1 | 0 | 0 | $\frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})$ | $\sqrt{2}\,\mathsf{t}_3$ |
| $\pi^-$ | $\rho^-$ | 1 | $-1$ | 0 | $d\bar{u}$ | $\mathsf{t}_-$ |
| $K^+$ | $K^{\star+}$ | $1/2$ | $1/2$ | 1 | $u\bar{s}$ | $\mathsf{v}_+$ |
| $K^0$ | $K^{\star0}$ | $1/2$ | $-1/2$ | 1 | $d\bar{s}$ | $\mathsf{u}_+$ |
| $\bar{K}^0$ | $\bar{K}^{\star0}$ | $1/2$ | $1/2$ | $-1$ | $s\bar{d}$ | $\mathsf{u}_-$ |
| $K^-$ | $K^{\star-}$ | $1/2$ | $-1/2$ | $-1$ | $s\bar{u}$ | $\mathsf{v}_-$ |
| $\eta_8$ | $\omega_8$ | 0 | 0 | 0 | $\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})$ | $\sqrt{2}\,\mathsf{t}_8$ |
| $\eta_0$ | $\omega_0$ | 0 | 0 | 0 | $\frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$ | $\frac{1}{\sqrt{3}}\,\mathsf{1}$ |

TABLE 2.1: Normalized $SU(3)_F$ flavor wave functions for mesons.

The remaining two $I_3 = 0$, $Y = 0$ states are then constructed so that one is a singlet ($\eta_0 \sim \mathsf{1}$) and the remaining octet member $\eta_8 \sim \mathsf{t}_8$ is orthogonal to $\pi_0$ and $\eta_0$. The resulting octet and singlet wave functions are collected in Table 2.1.

Unfortunately the identification of Table 2.1 with physical states only works out in the limit of exact $SU(3)$ flavor symmetry. If it is broken due to unequal quark masses, the states in the multiplets will no longer be mass-degenerate and the $SU(3)$ Casimirs are no longer good quantum numbers. However, the third isospin component and the hypercharge are still conserved and commute with the Hamiltonian, so they can still be used to label the states. As a consequence, mesons that carry the same $I_3$ and $S$ are allowed to mix. This concerns for example the pseudoscalar mesons $\pi^0$, $\eta_8$ and $\eta_0$ and the vector mesons $\rho^0$, $\omega_8$ and $\omega_0$ which carry $I_3 = S = 0$: their flavor wave functions can mix with each other, and the mixed states are those that will appear in the physical spectrum. In principle this can be seen already from the flavor matrix elements of the quark mass matrix:

$$\langle \eta_0 \,|\, \mathsf{M} \,|\, \eta_8 \rangle \sim m_u + m_d - 2m_s\,, \qquad \langle \pi_0 \,|\, \mathsf{M} \,|\, \eta_8 \rangle \sim m_u - m_d\,, \qquad \text{etc.} \tag{2.101}$$

Because of $m_u \approx m_d$, isospin symmetry is still approximately realized and the flavor breaking mostly comes from the strange-quark mass. Hence, also the isospin $I$ (related the Casimir of $SU(2)$) is still a good quantum number, which leaves only a mixing for $\eta_8$ and $\eta_0$. If we call the flavor states generically $\psi_8$ and $\psi_0$ and the mixed ones $\psi$ and $\psi'$, we can define a mixing angle:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_0 \end{pmatrix} \xrightarrow{\text{ideal}} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_0 \end{pmatrix}. \tag{2.102}$$

In the case of 'ideal mixing' we have $\cos\theta = 1/\sqrt{3}$, which means

$$\psi = \tfrac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d})\,, \qquad \psi' = s\bar{s}\,. \tag{2.103}$$

| M | $I$ | $S$ | P ($0^{-+}$) | V ($1^{--}$) | A ($1^{+-}$) | S ($0^{++}$) | A' ($1^{++}$) | T ($2^{++}$) |
|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 0 | $\pi$ (0.14) | $\rho$ (0.78) | $b_1$ (1.23) | $a_0$ (0.98) | $a_1$ (1.23) | $a_2$ (1.32) |
| 8, 1 | 0 | 0 | $\eta$ (0.55) | $\omega$ (0.78) | $h_1$ (1.17) | $f_0$ (0.50) | $f_1$ (1.28) | $f_2$ (1.28) |
| 1, 8 | 0 | 0 | $\eta'$ (0.96) | $\phi$ (1.02) | $h_1'$ (1.39) | $f_0'$ (0.99) | $f_1'$ (1.42) | $f_2'$ (1.52) |
| 8 | $\frac{1}{2}$ | $\pm 1$ | $K$ (0.50) | $K^*$ (0.90) | $K_1$ (1.27) | $K_0^*$ (0.68) | $K_1$ (1.40) | $K_2^*$ (1.43) |
| 8 | 1 | 0 | $\pi$ (1.30) | $\rho$ (1.47) | $b_1$ | $a_0$ (1.47) | $a_1$ | $a_2$ |
| 8, 1 | 0 | 0 | $\eta$ (1.29) | $\omega$ (1.42) | $h_1$ | $f_0$ (1.37) | $f_1$ | $f_2$ |
| 1, 8 | 0 | 0 | $\eta'$ (1.41) | $\phi$ (1.68) | $h_1'$ | $f_0'$ (1.51) | $f_1'$ | $f_2'$ |
| 8 | $\frac{1}{2}$ | $\pm 1$ | $K$ (1.46) | $K^*$ (1.41) | $K_1$ | $K_0^*$ (1.43) | $K_1$ | $K_2^*$ |

TABLE 2.2: $SU(3)_F$ classification of meson resonances (PDG 2012) in terms of $J^{PC}$, isospin $I$ and strangeness $S$. The upper table shows the ground states and the bottom one the first radial excitations. The masses in brackets are given in GeV (central values without error bars). Grey or missing entries are experimentally not well enough determined or have several possible candidates. Mesons with $I = S = 0$ belonging to different multiplets can mix with each other, so in these cases there is no strict identification with flavor-octet (M$=$ 8) and singlet states (M$=$ 1). Note also that $C-$parity is only a good quantum number for neutral mesons.

**Vector mesons.** Let's have a look at Table 2.2 and compare our expectations with the experimental spectrum. Vector meson ground states are an 'ideal channel' in several respects. We observe

$$m_\rho \approx m_\omega \qquad \text{and} \qquad m_\phi - m_{K^*} \approx m_{K^*} - m_\omega \,. \tag{2.104}$$

Suppose we have isospin symmetry ($m_u = m_d$) and ideal mixing, so that the $\omega$ is exclusively made of up and down quarks and the $\phi$ is a pure $s\bar{s}$ state. If the dynamics were such that the mass differences are entirely due to the different strange and up/down-quark masses, then we would expect from the flavor content alone:

$$\begin{aligned} m_\rho = m_\omega &= M_0 + 2m_u \,, \\ m_{K^*} &= M_0 + m_u + m_s \,, \\ m_\phi &= M_0 + 2m_s \,. \end{aligned} \tag{2.105}$$

$M_0$ is some flavor-independent mass that depends on $J^{PC}$ and the radial quantum number (otherwise we would have the same meson masses for each multiplet). Then Eq. (2.104) with the values from Table 2.2 yields $m_s - m_u \approx 120$ MeV, and we have the relation

$$m_\omega + m_\phi = 2m_{K^*} \,. \tag{2.106}$$

Had we replaced all masses in Eq. (2.105) by their squares, we would have instead obtained

$$m_\omega^2 + m_\phi^2 = 2m_{K^*}^2 \,. \tag{2.107}$$

Both versions are to a good extent realized in nature. (The idea behind this is that effective Hamiltonians depend on the masses either linearly or quadratically.) The determination of the mixing angle from the experimental values of $m_\omega$ and $m_\phi$ is somewhat ambiguous since one has to make assumptions about the dynamics of the system, and assume that the formula (2.107) or some analogue of it holds as well. Nevertheless, for vector mesons, ideal mixing is quite accurately realized as we can read off from Table 2.2. To a lesser extent the pattern is still visible in the $1^{+-}$, $1^{++}$ and $2^{++}$ channels because also the masses of $\{a, f\} \to K \to f'$ differ roughly by one unit of the strange-quark mass. On the other hand, there are two channels where the mass ordering does not work at all: the pseudoscalars $0^{-+}$ and the scalars $0^{++}$. Apparently there are further mechanisms at play to which we will turn now.

**No parity doublets.** If we set $m_u = m_d = m_s = 0$ we obtain a $SU(3)_V \times SU(3)_A$ chiral symmetry. In that case all mesons within a given $J^{PC}$ multiplet would become mass-degenerate. In addition, we would expect parity doublets for mesons with same $J$ but different $P$: if $|\lambda\rangle$ is an eigenstate of the Hamiltonian with positive parity, then the negative-parity state[8] $Q_a^A\,|\lambda\rangle$ is also an eigenstate with the same mass because the axial charge $Q_a^A$ commutes with $H$ in the chiral limit. From chiral symmetry we should therefore expect that the masses of the pseudoscalars are roughly degenerate with the scalars, that the vector mesons are degenerate with the axialvectors, and so on. Now, chiral symmetry is broken explicitly because $m_s \gg m_u \approx m_d \sim 0$, but we should still see remnants of this pattern in the spectrum. We don't: the pion is almost massless, and the vector mesons are much lighter than the axialvector ones.

On the other hand, the fact that $SU(2)_V$ still works out quite well (mesons with same isospin $I$ have roughly the same mass) tells us that something is wrong with the $SU(N_f)_A$ part. Combined with the unnaturally light pseudoscalar mesons, these are the symptoms of a *spontaneous* breaking of $SU(N_f)_A$, which would produce $N_f^2 - 1$ massless Goldstone bosons in the chiral limit. The three pions are indeed almost massless ($m_\pi \approx 140$ MeV); the kaons are still comparatively light, but they contain one strange quark so they will feel the impact of explicit chiral symmetry breaking more strongly than the pions.

**$\eta - \eta'$ mixing.** If chiral symmetry is indeed broken spontaneously, it should break all axial symmetries, $SU(3)_A$ and $U(1)_A$. Hence we would expect nine Goldstone bosons, including the pions, the kaons and both $\eta$ and $\eta'$ mesons. For ideal mixing, $\eta$ and $\eta'$ would be the analogues of $\omega$ and $\phi$ in the vector channel, with the flavor wave functions from Eq. (2.103). The $\eta$ would be mass-degenerate with the pion, and the $\eta'$ as a pure $s\bar{s}$ state would acquire mass due to the explicit chiral symmetry breaking, similarly to the kaons. If we assume that the formula (2.107) still holds for pseudoscalar mesons, $m_\eta^2 + m_{\eta'}^2 = 2m_K^2$, and set $m_\eta = m_\pi$, we expect $m_{\eta'} = 0.69$ GeV. If we now allow for an 'unmixing' of $\eta$ and $\eta'$ away from the ideal case, we might expect both masses to go in opposite directions: if $m_\eta > m_\pi$, we should have $m_{\eta'} < 0.69$ GeV. This is not realized at all: the $\eta$ is heavier than the kaon and the $\eta'$ mass is almost twice as large.

[8] That the axial charge switches sign under parity ($U_P\, Q^A\, U_P^{-1} = -Q^A$) can be derived along the same lines as in Eq. (2.96). If $U_P|\lambda\rangle = +|\lambda\rangle$, then $U_P\, Q^A|\lambda\rangle = -Q^A|\lambda\rangle$.

This argument and other ones lead us to believe that the $U(1)_A$ symmetry might not have been realized to begin with. It is satisfied by the classical QCD Lagrangian, but there is still the possibility of an anomalous symmetry breaking at the quantum level, where the classical symmetry does not survive the quantization. We have already anticipated in Eq. (2.29) that the divergence of the axial current picks up an extra term. Indeed, if we work out Eq. (2.79) in the isosinglet case we obtain

$$f_{\eta_0} m_{\eta_0}^2 = 2\,\frac{m_u + m_d + m_s}{3}\, r_{\eta_0} + \langle 0|\,\frac{g^2 N_f}{32\pi^2}\,\widetilde{F}_a^{\mu\nu}(0)\,F^a_{\mu\nu}(0)\,|\eta_0\rangle\,. \tag{2.108}$$

The term on the right-hand side is related to the topological susceptibility and remains present when the quark masses vanish, and therefore also the $\eta_0$ remains massive in the chiral limit. Through a mixing with $\eta_8$ the topological susceptibility will contribute to both $\eta$ and $\eta'$ masses.

**Missing exotics.** Another feature of Table (2.2) is that not all expected multiplets appear. The quantum numbers $0^{--}$, $0^{+-}$, $1^{-+}$ and $2^{+-}$, which are called 'exotic', are absent in the light-meson spectrum (although we do see some of these states higher up in the spectrum, for example the $\pi_1(1400)$ and $\pi_1(1600)$ which carry $1^{-+}$). The absence of exotic mesons can be understood from the nonrelativistic quark model. So far we have labeled $q\bar{q}$ states according to their $J^{PC}$ eigenvalues. Now assume that the total spin $S$ of the $q\bar{q}$ pair ($S=0$ or $S=1$) and its intrinsic orbital angular momentum $L = 0, 1, 2, \dots$ are also good quantum numbers. Then from the angular-momentum addition rules we have $|L-S| \leq J \leq L+S$, and we can motivate the following two relations: [9]

$$P = (-1)^{L+1} \qquad \text{and} \qquad C = (-1)^{L+S}\,. \tag{2.109}$$

The first one has the consequence that states with alternating $L$ have also alternating parity. The second one entails that once $L$ and $S$ are specified, $C$ (and therefore $J^{PC}$) is fixed as well.

These rules are quite efficient for cataloguing the possible $J^{PC}$ combinations:

- $L=0$ are orbital ground states ($s$ waves) and should therefore correspond to the lightest mesons. According to Eq. (2.109) they must have negative parity. $S=0$ gives us the pseudoscalars $0^{-+}$; from $S=1$ we obtain the vector mesons $1^{--}$.

- $L=1$ are orbital excitations ($p$ waves) with positive parity. From $S=0$ we obtain the axialvectors $1^{+-}$; from $S=1$ we get scalar ($0^{++}$), axialvector ($1^{++}$) and tensor mesons ($2^{++}$).

- From $L=2$ ($d$ waves) we obtain further vectors ($1^{--}$) plus states with $J=2$ and $J=3$, and so on for higher $L$.

[9] A $q\bar{q}$ pair has intrinsic parity $-1$ and its spatial wave function has parity $(-1)^L$; parity is multiplicative, hence the factor $(-1)^{L+1}$. Concerning the second equation: after $q$ and $\bar{q}$ were reversed by the charge conjugation, we bring them back to their original position. The anticommutativity of the quark fields gives a minus sign, a factor $(-1)^L$ comes from the parity of the relative motion, and a factor $(-1)^{S+1}$ from the spin since the spin state is symmetric for $S=1$ and antisymmetric for $S=0$.

This construction tells us that the masses of the $J^{PC}$ multiplets should increase in the same order as the columns displayed in Table (2.2):

$$\underbrace{\begin{bmatrix} \text{P} \\ 0^{-+} \\ {\scriptstyle S=0} \end{bmatrix} < \begin{bmatrix} \text{V} \\ 1^{--} \\ {\scriptstyle S=1} \end{bmatrix}}_{L=0} \quad < \quad \underbrace{\begin{bmatrix} \text{A} \\ 1^{+-} \\ {\scriptstyle S=0} \end{bmatrix} \lesssim \begin{bmatrix} \text{S} & \text{A'} & \text{T} \\ 0^{++} & 1^{++} & 2^{++} \\ & {\scriptstyle S=1} & \end{bmatrix}}_{L=1} . \tag{2.110}$$

For example, pseudoscalars and vectors are the lightest mesons because they are in an orbital $s$ wave. Since they carry different quark spin $S$, their mass splitting is generated by spin-spin interactions between quark and antiquark. This is called 'hyperfine splitting' because of its analogy to the hydrogen atom where the hyperfine structure is caused by the coupling between electron and proton spin. All other mesons are orbitally excited because they carry higher $L$. Notably, the analysis forbids exotic quantum numbers $0^{--}$, $0^{+-}$, $1^{-+}$ and $2^{+-}$. Possible exotic candidates are thus believed to be non-$q\bar{q}$ states, for example hybrids (states with valence quarks *and* gluons) or glueballs (states with valence gluons only).

It should be noted that the relations (2.109) are nonrelativistic because $P$ and $C$ are conserved quantum numbers whereas $L$ and $S$ are not. Only $J$ corresponds to a Casimir operator of the Poincaré group; $S$ and $L$ are not Poincaré-invariant and can mix in different reference frames. Therefore only $J^{PC}$ should be used to label multiplets. For example, already the pion's bound-state wave function in Eq. (2.80) contains $L = 1$ components, namely the structures $\not{q}$ and $[\not{q}, \not{P}]$ that depend on the relative momentum $q$ and therefore correspond to $p$ waves in the pion's rest frame. Similarly, exotic mesons are not generally forbidden as $q\bar{q}$ states but merely do not survive the non-relativistic treatment.

**Scalar mesons.** Another oddity in the multiplet arrangement of Table (2.2) concerns the nature of the scalar $0^{++}$ mesons. The mass ordering within the multiplet is not only far from 'ideal' but even reversed: the isosinglet $f_0(500)$ or $\sigma$ meson is the lightest, followed by a relatively light 'kaon' and almost degenerate $\{a_0, f_0'\}$. Hence, the ordering is $f \to K \to \{a, f'\}$, which doesn't make much sense given the contained flavor content: why would $a$ and $f'$ be mass-degenerate? Unless the mixing is ideal, in which case the $\sigma$ would be the $s\bar{s}$ state and should be the *heaviest* and not the lightest one.

Another argument comes from the quark-model analysis above: scalar mesons are $p$ waves and should have similar masses as their $1^{++}$ and $2^{++}$ counterparts. Interestingly, such a feature would be (roughly!) realized if we simply *removed* the scalar ground-state multiplet from the spectrum and identified the first radial excitations, the $a_0(1.47)$ and its partners, instead with the ground states. Such arguments initiated the idea (Jaffe, 1977) that the members of the $0^{++}$ ground-state nonet might not be actual mesons at all but rather *tetraquark* states: a group-theoretical analysis shows that such $qq\bar{q}\bar{q}$ states would again form a nonet, but now with the correct mass ordering (for example, the $\sigma$ would contain four light quarks and hence be the lightest state). This is just one possible explanation; since the scalar channel carries the quantum numbers of the vacuum it will also be sensitive to other effects, for example mixing with non-$q\bar{q}$ $0^{++}$ states such as glueballs.

### 2.3.2 Light baryons

**Baryon multiplets.** Let's leave the meson sector and carry on with baryons. Baryons are fermions, so their angular momentum takes half-integer values: $J^P = 1/2^\pm$, $3/2^\pm$, $5/2^\pm$, and so on. Each $J^P$ channel can in principle contain $SU(3)$ flavor octets, decuplets and singlets, with ground states and radial excitations. As before, $SU(3)$ flavor breaking entails that baryons with same isospin $I$ and strangeness $S$ quantum numbers can mix. The presently (well-)known baryons are collected in Table 2.4.

The permutation symmetry of three identical quarks gives us a powerful tool to constrain the structure of baryon wave functions. Baryons must satisfy the Pauli principle, i.e., in the flavor-symmetric limit their total wave function

$$\psi = \text{Dirac} \times \text{Flavor} \times \text{Color} \tag{2.111}$$

must be totally antisymmetric under exchange of any two quarks. 'Dirac' is here a shorthand for the full spatial and spin (or momentum and spin) contribution that transforms under the Poincaré group. We can then arrange each part in irreducible representations of the permutation group $S_3$, with definite symmetry, and figure out in the end which symmetry states are allowed in the combination. The color part transforms under $SU(3)_C$, and the singlet representation in $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\dots$ is totally antisymmetric (see Appendix A.2). If we use the quark basis states from Eq. (2.97) for the color part, the color wave function is given by $\varepsilon_{ijk}$. Hence, the remainder must be fully symmetric. The flavor wave functions transform under $SU(3)_F$, so let's discuss them next.

**Flavor wave functions for baryons.** Suppose $u$, $d$ and $s$ denote the three flavor vectors for the quarks that transform under the fundamental representation of $SU(3)$, for example in the basis (2.97). Combining three of them gives us in total $3\times3\times3 = 27$ possible flavor combinations which would transform under the 27-dimensional reducible representation of $SU(3)$. The irreducible representations contained in $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}$ differ by their symmetry, and to find out what they look like our goal is to classify them into *simultaneous* irreducible representations of $SU(3)$ and the permutation group $S_3$.

Take the generic combination $abc$, where $a$, $b$, $c$ can stand for any vector $u$, $d$, or $s$. There are six possible permutations of $abc$ which we obtain from applying the permutation operators

$$\begin{array}{cccccc} 1 & P_{12} & P_{13} & P_{23} & P_{23}\,P_{12} & P_{13}\,P_{12}\,. \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ abc & bac & cba & acb & bca & cab \end{array} \tag{2.112}$$

These are the group elements of $S_3$ that can be visualized by the Cayley graph in Fig. 2.6. Any permutation can be reconstructed from a transposition $P_{12}$ and a cyclic permutation $P_{123}$ (which act on the *positions* of $a$, $b$, $c$) via paths along the Cayley graph:

$$P_{13} = P_{12}\,P_{123}\,, \quad P_{23} = P_{12}\,P_{123}^2\,, \quad P_{23}\,P_{12} = P_{123}\,, \quad P_{13}\,P_{12} = P_{123}^2\,. \tag{2.113}$$

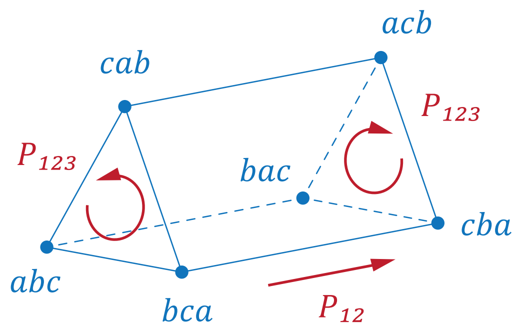


FIGURE 2.6: Cayley graph for the permutation group $S_3$. Any permutation can be reconstructed from a transposition $P_{12}$ and a cycle $P_{123}$.

Let's start from the following six combinations:

$$\psi_1^\pm = \frac{ab \pm ba}{2}\, c\,, \qquad \psi_2^\pm = \frac{bc \pm cb}{2}\, a\,, \qquad \psi_3^\pm = \frac{ca \pm ac}{2}\, b\,. \tag{2.114}$$

They are symmetric or antisymmetric under a transposition $P_{12}$. We can rearrange them into a symmetric singlet $\mathcal{S}$, an antisymmetric singlet $\mathcal{A}$, and two doublets $\mathcal{D}_1$ and $\mathcal{D}_2$ whose upper (lower) entries are antisymmetric (symmetric) under transpositions:

$$\begin{gathered} \mathcal{S} = \psi_1^+ + \psi_2^+ + \psi_3^+\,, \\ \mathcal{A} = \psi_1^- + \psi_2^- + \psi_3^-\,, \\ \mathcal{D}_1 = \begin{pmatrix} \psi_2^- - \psi_3^- \\ -\frac{1}{\sqrt{3}}\left(\psi_2^+ + \psi_3^+ - 2\psi_1^+\right) \end{pmatrix}, \qquad \mathcal{D}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}}\left(\psi_2^- + \psi_3^- - 2\psi_1^-\right) \\ \psi_2^+ - \psi_3^+ \end{pmatrix}. \end{gathered} \tag{2.115}$$

They transform under irreducible representations of $S_3$ because they do not mix under any permutation, and thereby they generate three invariant subspaces:

$$P_{12}\begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathcal{D}_i \end{bmatrix} = \begin{bmatrix} \mathcal{S} \\ -\mathcal{A} \\ \mathsf{M}_{12}\,\mathcal{D}_i \end{bmatrix}, \quad P_{123}\begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathcal{D}_i \end{bmatrix} = \begin{bmatrix} \mathcal{S} \\ \mathcal{A} \\ \mathsf{M}_{123}\,\mathcal{D}_i \end{bmatrix}, \quad \mathsf{M}_{123} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

with $\mathsf{M}_{12} = \mathrm{diag}(-1, 1)$. $\mathcal{S}$ is totally symmetric and invariant under any permutation. $\mathcal{A}$ is totally antisymmetric under exchange of any two entries and therefore it picks up a minus sign under a transposition. Both subspaces are one-dimensional. The doublets form a two-dimensional subspace; they are constructed so that $\mathcal{D}_i \cdot \mathcal{D}_j$ is a singlet. They transform under the same matrix representations $\mathsf{M}_{12}$ and $\mathsf{M}_{123}$ from where all other ones can be reconstructed via Eq. (2.113). In the language of Young diagrams (which are discussed in detail in App. A.3) the multiplets are given by

$$\mathcal{S} = \begin{array}{|c|c|c|}\hline \phantom{x} & \phantom{x} & \phantom{x} \\ \hline\end{array}\,, \qquad \mathcal{D}_{1,2} = \begin{array}{|c|c|}\hline \phantom{x} & \phantom{x} \\ \hline \phantom{x} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \mathcal{A} = \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \phantom{x} \\ \hline\end{array}\,, \tag{2.116}$$

but Eqs. (2.114–2.115) are in fact all we need in the following.

| | $uuu$ | $uud$ | $ddu$ | $ddd$ | $uus$ | $uds$ | $dds$ | $ssu$ | $ssd$ | $sss$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{S}$ | $\Delta^{++}$ | $\Delta^{+}$ | $\Delta^{0}$ | $\Delta^{-}$ | $\Sigma^{+}$ | $\Sigma^{0}$ | $\Sigma^{-}$ | $\Xi^{0}$ | $\Xi^{-}$ | $\Omega^{-}$ |
| $\mathcal{D}_1$ | | $p$ | $n$ | | $\Sigma^{+}$ | $\Sigma^{0}$ | $\Sigma^{-}$ | $\Xi^{0}$ | $\Xi^{-}$ | |
| $\mathcal{D}_2$ | | | | | | $\Lambda^{0}$ | | | | |
| $\mathcal{A}$ | | | | | | $\Lambda^{0}$ | | | | |

TABLE 2.3: $SU(3)_F$ flavor wave functions for baryons: use Eqs. (2.114)–(2.115) to work them out explicitly.

Let's apply it to construct the flavor wave function for a baryon with flavor content $uud$, such as the proton or the $\Delta^+$. If we take $a = b = u$ and $c = d$, then we have from Eq. (2.114):

$$\psi_1^+ = uud\,, \qquad \psi_1^- = 0\,, \qquad \psi_2^\pm = \pm\psi_3^\pm = \frac{ud \pm du}{2}\,u\,, \tag{2.117}$$

and therefore

$$\begin{aligned} \mathcal{S}(uud) &= uud + udu + duu\,, \\ \mathcal{D}_1(uud) &= \begin{pmatrix} udu - duu \\ -\frac{1}{\sqrt{3}}\,(udu + duu - 2uud) \end{pmatrix}, \end{aligned} \tag{2.118}$$

together with $\mathcal{D}_2(uud) = \mathcal{A}(uud) = 0$. Apart from overall normalization, $\mathcal{S}(uud)$ is the flavor wave function of the $\Delta^+$ and $\mathcal{D}_1(uud)$ is that of the proton. Had we started from $ddu$ instead of $uud$, we would have obtained the wave functions for the $\Delta^0$ and the neutron (replace $u \leftrightarrow d$ in the equation above). The combination $uuu$ returns only a singlet ($\Delta^{++}$), and from $uds$ we get everything: $\mathcal{S}$, $\mathcal{A}$ and two doublets. If we take all 10 combinations with different flavor content into account ($uuu$, $ddd$, $sss$, $uud$, $uus$, $ddu$, $dds$, $ssu$, $ssd$, $uds$), the permutation group gives us

- ten symmetric singlets, which form the flavor decuplet with $\Delta$, $\Sigma$, $\Xi$ and $\Omega$,
- eight doublets that form the flavor octet, including proton, neutron, $\Sigma$, $\Xi$ and $\Lambda$,
- and one antisymmetric singlet from $uds$, the flavor singlet for $\Lambda$.

These are just the irreducible representations of $SU(3)_F$: decuplet, octet and singlet. We have successfully rearranged the baryon flavor wave functions so that they transform under irreducible representations of both $SU(3)_F$ and $S_3$. The result is collected in Table 2.3. Including charm as a fourth flavor, we can immediately extend the construction to $SU(4)_F$ which would give us 20 symmetric singlets, 20 doublets and 4 antisymmetric singlets:

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{20}_S \oplus \mathbf{20}_{M_A} \oplus \mathbf{20}_{M_S} \oplus \mathbf{4}_A\,. \tag{2.119}$$

In the $SU(2)_F$ case, on the other hand, we get four symmetric singlets (the four $\Delta$ baryons) and two doublets (proton and neutron):

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_S \oplus \mathbf{2}_{M_A} \oplus \mathbf{2}_{M_S}\,. \tag{2.120}$$

**Full baryon wave function.** The remaining question is what the Dirac part in Eq. (2.111) looks like. From the above analysis we conclude that even without knowing its explicit form, it will have the same permutation-group structure. Therefore, we can also collect it in $\mathcal{S}$, $\mathcal{A}$ and two doublets $\mathcal{D}_1$, $\mathcal{D}_2$ and rewrite (2.111) as

$$\psi = \{\mathcal{S},\,\mathcal{A},\,\mathcal{D}_1,\,\mathcal{D}_2\}_D \times \{\mathcal{S},\,\mathcal{A},\,\mathcal{D}_1,\,\mathcal{D}_2\}_F \times \mathcal{A}_C \stackrel{!}{=} \mathcal{A}_{\text{total}}\,. \tag{2.121}$$

The Dirac-flavor part must be symmetric. How can we construct *direct* products of $S_3$ multiplets? The product of two symmetric or antisymmetric singlets is obviously again a symmetric singlet: $\mathcal{S}\,\mathcal{S}' = \mathcal{S}''$, $\mathcal{A}\,\mathcal{A}' = \mathcal{S}'''$. The inner product of two doublets is also symmetric, as one can infer from the orthogonality of the representation matrices $\mathsf{M}_{12}$ and $\mathsf{M}_{123}$:

$$(\mathsf{M}\,\mathcal{D})\cdot(\mathsf{M}\,\mathcal{D}') = \mathcal{D}\,\mathsf{M}^T\mathsf{M}\,\mathcal{D}' = \mathcal{D}\cdot\mathcal{D}'\,. \tag{2.122}$$

It is a symmetric singlet because it is invariant under any permutation. Let's denote doublets generically by

$$\mathcal{D} = \begin{pmatrix} a \\ s \end{pmatrix}, \tag{2.123}$$

with a mixed-antisymmetric upper entry $a$ and a mixed-symmetric lower component $s$. We can combine two doublets also to an antisymmetric singlet or to another doublet. Here are all possible combinations:

$$\begin{array}{lllll} \mathcal{S}'': & \mathcal{D}\cdot\mathcal{D}' & := aa' + ss'\,, & \mathcal{S}\mathcal{S}'\,, & \mathcal{A}\mathcal{A}'\,, \\ \mathcal{A}'': & \mathcal{D}\times\mathcal{D}' & := as' - sa'\,, & \mathcal{S}\mathcal{A}\,, & \\ \mathcal{D}'': & \mathcal{D}*\mathcal{D}' & := \begin{pmatrix} as' + sa' \\ aa' - ss' \end{pmatrix}, & \mathcal{S}\mathcal{D}\,, & \mathcal{A}\times\mathcal{D} := \mathcal{A}\begin{pmatrix} s \\ -a \end{pmatrix}. \end{array} \tag{2.124}$$

We have simply defined the operations $\times$ and $*$ to fit our purposes. You can verify the validity of these relations explicitly: the resulting singlets $\mathcal{S}''$ stay invariant under permutations, the antisymmetric singlets $\mathcal{A}''$ pick up a minus sign for an odd permutation, and the doublets $\mathcal{D}''$ transform according to (2.3.2). In summary, we see that in order to obtain a fully antisymmetric total wave function for the baryon, the Dirac and flavor parts must have the same permutation-group symmetry:

$$\mathcal{A}_{\text{total}} = \begin{cases} (\mathcal{D}_D\cdot\mathcal{D}_F)\,\mathcal{A}_C & \text{(octet)}\,, \\ (\mathcal{S}_D\,\mathcal{S}_F)\,\mathcal{A}_C & \text{(decuplet)}\,, \\ (\mathcal{A}_D\,\mathcal{A}_F)\,\mathcal{A}_C & \text{(singlet)}\,. \end{cases} \tag{2.125}$$

**Quark model.** What will the contributions $\mathcal{D}_D$, $\mathcal{S}_D$, $\mathcal{A}_D$ look like? Dynamically, we expect states that appear in orbital $s$ waves ($L = 0$) to be the lightest ones. In analogy to Eq. (2.109), the nonrelativistic quark model gives us the parity assignment $P = (-1)^L$, i.e. all positive-parity states have even orbital angular momentum and for all negative-parity states $L$ is odd. The combination of three spins $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}$ in a three-quark $qqq$ state only permits total spin $S = \frac{1}{2}$ or $\frac{3}{2}$. With the angular momentum addition rule $J = |L-S|, \dots L+S$ the only possible $s-$waves ($L = 0$) can therefore appear in $J = \frac{1}{2}^+$ and $\frac{3}{2}^+$.

| M | $I$ | $S$ | $\frac{1}{2}^+$ | $\frac{3}{2}^+$ | $\frac{5}{2}^+$ | $\frac{1}{2}^-$ | $\frac{3}{2}^-$ | $\frac{5}{2}^-$ |
|---|---|---|---|---|---|---|---|---|
| 8 | $\frac{1}{2}$ | 0 | $\mathbf{N(0.94)}$<br>$N(1.44)$<br>$N(1.71)$ | $N(1.72)$<br>$N(1.90)$ | $N(1.68)$ | $N(1.54)$<br>$N(1.65)$ | $N(1.52)$<br>$N(1.70)$<br>$N(1.88)$ | $N(1.68)$ |
| 10 | $\frac{3}{2}$ | 0 | $\Delta(1.91)$ | $\boldsymbol{\Delta(1.23)}$<br>$\Delta(1.60)$<br>$\Delta(1.92)$ | $\Delta(1.91)$ | $\Delta(1.62)$ | $\Delta(1.70)$ | $\Delta(1.93)$ |
| 8 | 0 | $-1$ | $\boldsymbol{\Lambda(1.12)}$<br>$\Lambda(1.60)$ | $\Lambda(1.89)$ | $\Lambda(1.82)$<br>$\Lambda(2.11)$ | $\Lambda(1.67)$<br>$\Lambda(1.80)$ | $\Lambda(1.69)$ | $\Lambda(1.83)$ |
| 1 | 0 | $-1$ | $\Lambda(1.81)$ | | | $\Lambda(1.41)$ | $\Lambda(1.52)$ | |
| 8 | 1 | $-1$ | $\boldsymbol{\Sigma(1.19)}$<br>$\Sigma(1.66)$ | | $\Sigma(1.92)$ | $\Sigma(1.75)$ | $\Sigma(1.67)$ | $\Sigma(1.78)$ |
| 10 | 1 | $-1$ | | $\boldsymbol{\Sigma(1.39)}$ | | | $\Sigma(1.94)$ | |
| 8 | $\frac{1}{2}$ | $-2$ | $\boldsymbol{\Xi(1.31)}$ | | | $\Xi(1.69)$ | $\Xi(1.82)$ | |
| 10 | $\frac{1}{2}$ | $-2$ | | $\boldsymbol{\Xi(1.53)}$ | | | | |
| 10 | 0 | $-3$ | | $\boldsymbol{\Omega(1.67)}$ | | | | |

TABLE 2.4: $SU(3)_F$ classification of known baryons in terms of $J^P$, isospin $I$ and strangeness $S$. Only well-established states (three and four-star resonances, PDG 2012) are included, with masses in GeV. The table includes both ground states and excitations. Similarly to the singlet-octet mixing in the meson sector, baryons with same $I$ and $S$ quantum numbers can mix among the multiplets. In these cases the assignment above is based on quark-model expectations. The bold entries show the $s-$wave ground states according to the quark model.

In the quark model, the Dirac parts are taken to be the direct product of $O(3)$ orbital and $SU(2)$ spin wave functions. The latter have the same form as in Eq. (2.118) and the first column in Table 2.3 (replace $u$ by $\uparrow$ and $d$ by $\downarrow$), which yields four permutation-group singlets $\mathcal{S}$ with spin $S = 3/2$ and two doublets $\mathcal{D}$ with spin $S = 1/2$. For orbital ground states the orbital wave functions are spatially symmetric, so the only possible Dirac states in (2.125) are $\mathcal{D}_D$ and $\mathcal{S}_D$. In that case we also have $L = 0 \Rightarrow J = S$ and therefore the only possible combinations are

$$L = 0: \qquad \mathcal{A}_{\text{total}} = \begin{cases} (\mathcal{D}_D \cdot \mathcal{D}_F)\, \mathcal{A}_C & (J = \frac{1}{2}^+, \text{ octet}), \\ (\mathcal{S}_D\, \mathcal{S}_F)\, \mathcal{A}_C & (J = \frac{3}{2}^+, \text{ decuplet}). \end{cases} \tag{2.126}$$

Hence, the flavor octet baryons correspond to $J = \frac{1}{2}^+$ and the decuplet baryons to $\frac{3}{2}^+$. There is no flavor-singlet baryon $\Lambda^0$, at least it cannot appear in an orbital ground state. These ground states are highlighted in Table 2.4.

Higher $L$ states can be constructed like in the meson case: $L = 1$ entails $P = -1$, and in combination with $S = \frac{1}{2}$ or $\frac{3}{2}$ this gives us the negative-parity baryons with $J^P = \frac{1}{2}^-$, $\frac{3}{2}^-$ and $\frac{5}{2}^-$. In principle one can also directly combine the $SU(3)$ flavor multiplets $\mathcal{D}_F$, $\mathcal{S}_F$ and $\mathcal{A}_F$ with the $SU(2)$ spin multiplets $\mathcal{D}_D$ and $\mathcal{S}_D$. In that way one arrives at the '$SU(6)$' quark-model classification in terms of

- 56 symmetric singlets, which are relevant for orbital ground states; from Eq. (2.125): $2 \cdot 8 + 4 \cdot 10 + 0 \cdot 1 = 56$,
- 20 antisymmetric singlets from the second row in (2.124),
- and 70 doublets; third row in (2.124).

One should keep in mind, however, that these multiplets do not count the actual number of particles in the spectrum but only the wave functions for each spin polarization. They are subsequently combined with appropriate spatial wave functions times the color part so that the total baryon wave functions are antisymmetric.

A final remark: the spin-statistics theorem also motivated the introduction of the color degree of freedom. The $\Delta^{++}$ carries three up quarks ($uuu$) and has all spins aligned ($\uparrow\uparrow\uparrow$), which does not yield a totally antisymmetric wave function. If we wanted to respect the Pauli principle *without* color, then Eq. (2.124) provides us with the following options:

$$\mathcal{A}_{\text{total}} = \begin{cases} \mathcal{D}_D \times \mathcal{D}_F & \text{(octet)}\,, \\ \mathcal{A}_D \, \mathcal{S}_F & \text{(decuplet)}\,, \\ \mathcal{S}_D \, \mathcal{A}_F & \text{(singlet)}\,. \end{cases} \tag{2.127}$$

With the $SU(2)$ spin wave functions $\mathcal{D}_D$ and $\mathcal{S}_D$ (but no $\mathcal{A}_D$) we could still construct nucleons but not $\Delta$ baryons, at least not in orbital ground states.