2.4 Spontaneous chiral symmetry breaking

We have earlier seen that renormalization introduces a scale. Without a scale in the theory all hadrons would be massless (at least for vanishing quark masses), so the anomalous breaking of scale invariance is a necessary ingredient to understand their nature. The second ingredient is spontaneous chiral symmetry breaking. We will see that this mechanism plays a quite important role in the light hadron spectrum: it is not only responsible for the Goldstone nature of the pions, but also the origin of the 'constituent quarks' which produce the typical hadronic mass scales of $\sim$1 GeV.

Spontaneous symmetry breaking. Let's start with some general considerations. Suppose φ_i are a set of (potentially composite) fields which transform nontrivially under some continuous global symmetry group G. For an infinitesimal transformation we have $\delta\varphi_i = i\varepsilon_a(\mathsf{t}_a)_{ij}\,\varphi_j$, where ε_a are the group parameters and t_a the generators of the Lie algebra of G in the representation to which the φ_i belong. Let's call the representation matrices $D_{ij}(\varepsilon) = \exp(i\varepsilon_a\mathsf{t}_a)_{ij}$. The quantum-field theoretical version of this relation is

$$e^{i\varepsilon_a Q_a}\,\varphi_i\,e^{-i\varepsilon_a Q_a} = D^{-1}_{ij}(\varepsilon)\,\varphi_j \qquad \Leftrightarrow \qquad [Q_a,\varphi_i] = -(\mathsf{t}_a)_{ij}\,\varphi_j\,, \tag{2.128}$$

where the charge operators Q_a form a representation of the algebra on the Hilbert space. (An example of this is Eq. (2.47).) If the symmetry group leaves the vacuum invariant, $e^{i\varepsilon_a Q_a}\,|0\rangle = |0\rangle$, then all generators Q_a must annihilate the vacuum: $Q_a|0\rangle = 0$. Hence, when we take the vacuum expectation value (VEV) of this equation we get

$$\langle 0\,|\varphi_i|\,0\rangle = D^{-1}_{ij}(\varepsilon)\,\langle 0\,|\varphi_j|\,0\rangle\,. \tag{2.129}$$

If the φ_i had been invariant under G to begin with, this relation would be trivially satisfied. Because they transform nontrivially, $D^{-1}_{ij}(\varepsilon)$ is not the identity matrix for all ε_a and so these vacuum expectation values must vanish:

$$Q_a\,|0\rangle = 0 \qquad \Rightarrow \qquad \langle 0\,|\varphi_i|\,0\rangle = 0\,. \tag{2.130}$$

This is the 'Wigner-Weyl realization' of a symmetry.

On the other hand, if an operator which is not invariant under G develops a nonzero vacuum expectation value, then the symmetry G is spontaneously broken. This is the 'Nambu-Goldstone realization' of the symmetry:

$$\langle 0|\,[Q_a,\varphi_i]\,|0\rangle = -(\mathsf{t}_a)_{ij}\,\langle 0|\varphi_j|0\rangle \neq 0\,. \tag{2.131}$$

Then one would conclude that the charges do not annihilate the vacuum: $Q_a|0\rangle \neq 0$. Since the symmetry is classically realized, they still commute with the Hamiltonian and we have found another energy-degenerate vacuum:

$$Q_a|0\rangle = |\eta\rangle \neq 0 \quad \Rightarrow \quad H|0\rangle = 0 \;\longrightarrow\; H|\eta\rangle = 0\,. \tag{2.132}$$

Unfortunately we have to be careful with these statements because in the case of spontaneous symmetry breaking the charges are not well defined. $|\eta\rangle$ is not a normalizable state, which we can see from using the definition of the charge (2.3) and translation invariance. However, commutators involving the charges can still be defined, so when discussing spontaneous symmetry breaking we should start from Eq. (2.131). The Goldstone theorem follows if we insert the completeness relation (2.71) in that equation:

$$\begin{aligned}
\langle 0|\left[Q_a(x_0),\varphi(0)\right]|0\rangle &= \int d^3x\,\langle 0|\left[j_a^0(x),\varphi(0)\right]|0\rangle \\
&= \sum_\lambda \int \frac{d^3p}{2E_p}\,\frac{1}{(2\pi)^3}\int d^3x\left(R_{a\lambda}(\boldsymbol{p})\,e^{-ipx} - R^\star_{a\lambda}(\boldsymbol{p})\,e^{ipx}\right) \\
&= \sum_\lambda \frac{R_{a\lambda}(0)\,e^{-im_\lambda x_0} - R^\star_{a\lambda}(0)\,e^{im_\lambda x_0}}{2m_\lambda} \stackrel{!}{=} \text{const.}
\end{aligned} \tag{2.133}$$

In going from the first to the second row we have used translation invariance (2.76) to factor out the phases $e^{\pm ipx}$, and we defined $R_{a\lambda}(\boldsymbol{p}) := \langle 0|\,j_a^0(0)\,|\lambda\rangle\langle\lambda|\,\varphi(0)\,|0\rangle$. The integral over x produces $\delta^3(\boldsymbol{p})$, so that $p_0 = E_p$ becomes m_λ. It is crucial that the VEV on the right-hand side is nonzero *and* time-independent. That requirement can only be met if for each charge Q_a there is a mode $|\lambda\rangle$ with

$$m_\lambda = 0 \qquad \text{and} \qquad \frac{R_{a\lambda}(0)}{im_\lambda} \neq 0 \quad \text{and real}. \tag{2.134}$$

(To see this, perform a Taylor expansion of the exponentials $e^{\pm im_\lambda x_0}$.) Thus, for each generator that does not leave the vacuum invariant there is a massless Goldstone boson whose vacuum overlap with j_a^0 and φ is non-zero. The other modes with $m_\lambda \neq 0$ (excited states) must have $R_{a\lambda}(0) = 0$.

Chiral condensate. So how does spontaneous breaking of chiral symmetry come about in QCD? First we have to identify potential candidates for vacuum condensates that break chiral symmetry. Earlier we talked about taking Dirac-flavor traces of the quark four-point function. What if we trace the quark propagator $S_{\alpha\beta}(x,x)$ itself? The result will be the vacuum expectation value of either of the currents in Eq. (2.12):

$$-\,\Gamma_{\beta\alpha}\,\mathsf{t}_a\,S_{\alpha\beta}(x,x) = \langle 0|j_a^\Gamma(x)|0\rangle = \langle 0|j_a^\Gamma(0)|0\rangle. \tag{2.135}$$

Because of translation invariance they cannot depend on x and must be (dimensionful) constants, which are consequently zero due to Lorentz and parity invariance. The only possible exception is the scalar condensate which carries the quantum numbers of the vacuum (0^{++}):

$$\langle 0|S(0)|0\rangle = \langle 0|\,\overline{\psi}(0)\psi(0)\,|0\rangle =: \langle\overline{\psi}\psi\rangle. \tag{2.136}$$

From Eq. (2.42) we infer $[Q_a^V, S_b(x)] = if_{abc}\,S_c(x)$. Therefore, if $SU(N_f)_V$ is unbroken and hence $Q_a^V|0\rangle = 0$, all non-singlet scalar condensates must vanish as well. The singlet condensate is then identical for all flavors:

$$\langle 0|S_a(0)|0\rangle = 0 \quad \Rightarrow \quad \langle\bar{u}u\rangle - \langle\bar{d}d\rangle = 0, \quad \langle\bar{u}u\rangle + \langle\bar{d}d\rangle - 2\langle\bar{s}s\rangle = 0, \tag{2.137}$$

and therefore $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \langle\bar{s}s\rangle = \langle\overline{\psi}\psi\rangle/3$. On the other hand, a scalar bilinear of two quark fields breaks chiral symmetry (that is, it breaks $SU(N_f)_A \times U(1)_A$), as we saw in Eq. (2.36), so in a chirally symmetric theory of massless quarks this quantity should also vanish. But does it?

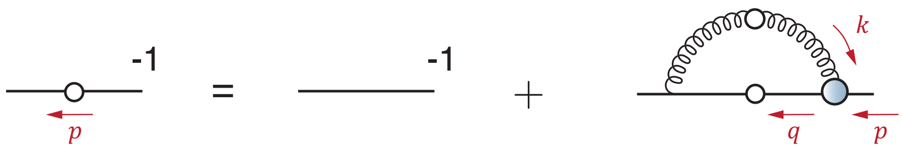

FIGURE 2.7: Quark DSE.

Quark mass function. Since the condensate is the trace of the quark propagator, let's have a closer look at the propagator itself. Take the inverse quark propagator from the Lagrangian:[10]

$$S^{-1}(p) = A(p^2)\,\big(i\not{p} + M(p^2)\big) = Z_2(i\not{p} + Z_m m) + \Sigma(p)\,. \qquad (2.139)$$

The first equality is the general form from Poincaré covariance and parity invariance; i.e., there cannot be more terms than those two. $M(p^2)$ is the quark *mass function* since it perturbatively reduces to the tree-level mass in the Lagrangian. From taking the inverse we obtain the general form for the quark propagator itself:

$$S(p) = \frac{-i\not{p} + M(p^2)}{(p^2 + M(p^2)^2)\,A(p^2)}\,. \qquad (2.140)$$

The second equality in (2.139) is the quark's Dyson-Schwinger equation, where the right-hand side contains the tree-level part plus self-energy quantum corrections, see Fig. 2.7. The trace with $\mathbb{1}$ singles out the contribution from the mass function so that the condensate is proportional to the integrated quark mass function:[11]

$$-\langle \bar{u}u\rangle = N_C \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\,S(p) = \frac{N_C}{(2\pi)^2}\int dp^2\,\frac{p^2}{A(p^2)}\,\frac{M(p^2)}{p^2+M(p^2)^2}\,. \qquad (2.141)$$

Therefore, for a chirally symmetric Lagrangian ($m = 0$), the quark mass function should be zero for all p^2 which means that the resulting quark propagator is chirally symmetric: $\{\gamma_5, S(p)\} = 0$. Indeed, this is what we find if we evaluate the self-energy order by order in perturbation theory: each diagram contains an odd number of massless tree-level propagators ($\sim \not{p}$) and vertices ($\sim \gamma^\mu$), and we can never generate a scalar term because the trace of an odd product of γ-matrices is zero. Hence, the resulting mass function will be zero to all orders in perturbation theory. Can we obtain a non-zero mass function *nonperturbatively*?

[10]We temporarily switch to Euclidean conventions:

$$a_E^\mu = (\boldsymbol{a}, ia_0)\,, \quad \gamma_E^\mu = (-i\boldsymbol{\gamma}, \gamma_0)\,, \quad \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu} \quad \Rightarrow \quad a_E\cdot b_E = -a\cdot b\,, \quad \not{a}_E = i\not{a}\,. \qquad (2.138)$$

A scalar propagator $i/(p^2-m^2+i\varepsilon)$ becomes $1/(p_E^2+m^2)$ if the global factor i is removed, and positive p_E^2 means spacelike. We drop the label 'E'.

[11]With hyperspherical coordinates one obtains $\int d^4p\, f(p^2) = \pi^2 \int dp^2 p^2\, f(p^2)$; an additional factor 4 comes from the Dirac trace over the unit matrix. The condensate renormalizes like the mass term in the Lagrangian, i.e., the right-hand side picks up an additional factor $Z_2 Z_m$ so that the product $m\langle\bar\psi\psi\rangle$ is renormalization-point independent.

Munczek-Nemirovsky model. The answer is yes; in fact, this feature can be illustrated already in simple models for the DSE. Let's assume that the quark-gluon vertex in the self-energy remains at tree-level, and only the internal quark and gluon propagators are dressed ('rainbow truncation'). In Feynman gauge the gluon is diagonal in its Lorentz indices, so we can write the self-energy as

$$\Sigma(p) = \int d^4k \, \gamma^\mu \, S(p+k) \, \gamma^\mu \, D(k) \,, \tag{2.142}$$

where $D(k)$ is proportional to the gluon propagator. It must be a scalar function of the gluon momentum k^2 with mass dimension -2. At large k^2 it must also be proportional to QCD's running coupling, $D(k^2) \sim \alpha_s(k^2)/k^2$, because there the gluon propagator and quark-gluon vertex approach their tree-level values. On the other hand, since the coupling becomes large in the infrared, calculating the self-energy allows us to test the impact of a nonperturbative large coupling on the quark propagator. For example, we can employ the *Munczek-Nemirovsky model* where the gluon propagator is just a δ-function peaked at the origin, equipped with some mass scale Λ:

$$D(k) \to \Lambda^2 \, \delta^4(k) \,. \tag{2.143}$$

Since the self-energy can be integrated analytically, this model is UV-finite and instead of imposing renormalization conditions we can set all renormalization constants to 1. The result for the self-energy is

$$S(p) = \frac{-i\not{p} + M}{(p^2+M^2)\,A} \quad \Rightarrow \quad \Sigma(p) = \Lambda^2 \, \gamma^\mu S(p) \gamma^\mu = \Lambda^2 \, \frac{2i\not{p} + 4M}{(p^2+M^2)\,A} \,, \tag{2.144}$$

where we suppressed the momentum dependencies of $A(p^2)$ and $M(p^2)$ for brevity. Inserting this in the DSE leads to selfconsistent equations for the two quark dressing functions:

$$A = 1 + \frac{2\Lambda^2}{(p^2+M^2)\,A} \,, \qquad AM = m + 2M \, \frac{2\Lambda^2}{(p^2+M^2)\,A} \,. \tag{2.145}$$

In the chiral limit ($m = 0$), we see from the second equation that the trivial solution $M = 0$ is always possible. It leads to a quadratic equation for A whose result is

$$M(p^2) = 0 \,, \qquad A(p^2) = \tfrac{1}{2} \left(1 + \sqrt{1 + 8\,\Lambda^2/p^2} \right) . \tag{2.146}$$

It has the correct perturbative behavior for $p^2 \to \infty$, namely $M = 0$ and $A \to 1$, so it reverts the quark propagator back to its tree-level form and preserves chiral symmetry. On the other hand, $A(p^2)$ diverges for $p^2 \to 0$, so this cannot be the whole story. Indeed there is now also another solution with $M \neq 0$:

$$M(p^2) = \sqrt{\Lambda^2 - p^2} \,, \qquad A(p^2) = 2 \,. \tag{2.147}$$

It breaks chiral symmetry spontaneously and is finite in the infrared. Both solutions are connected at the point $p^2 = \Lambda^2$, see Fig. 2.8. If we switch on a quark mass $m \neq 0$, the curves become smooth.

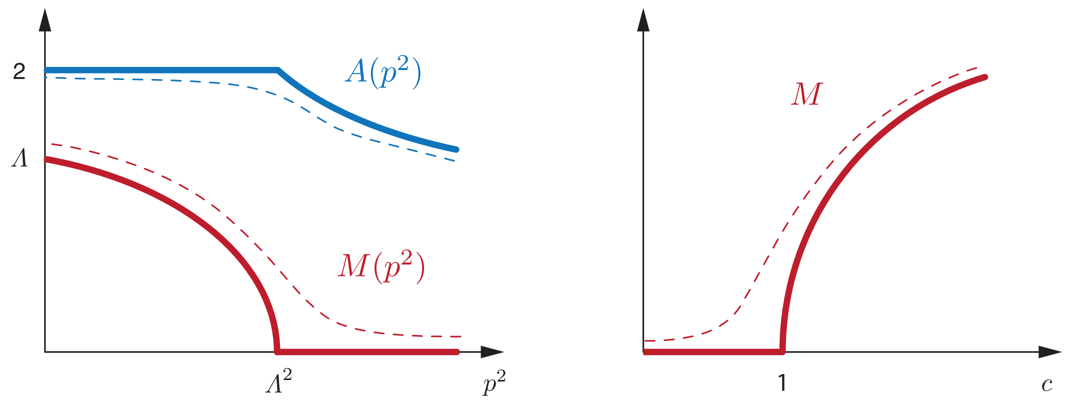

FIGURE 2.8: Quark propagator in the Munczek-Nemirovsky model (left) and NJL model (right). The dashed lines show the qualitative behavior of the solutions at $m \neq 0$.

Despite the simplicity of the model, these results capture already the essence of more realistic DSE calculations: coming from large momenta, the onset of the non-symmetric phase sets in at some typical hadronic scale Λ, below which a mass term is spontaneously generated. The mass function in the infrared defines the relevant quark mass at low momenta that is relevant for hadrons, so it can be viewed as a *constituent-quark* mass scale. Thus, the quark mass function encodes the transition from a current quark at large momenta to a constituent quark in the infrared. If we insert the combined solution in Eq. (2.141), the resulting quark condensate in the chiral limit becomes proportional to Λ^3. With $\Lambda = 1$ GeV we even get a reasonable value for the quark condensate:

$$-\langle \bar{u}u \rangle = \frac{2}{15} \frac{N_C}{(2\pi)^2} \Lambda^3 \quad \to \quad \sim (220\,\text{MeV})^3 . \tag{2.148}$$

Contact interaction. The shortcoming of the Munczek-Nemirovsky model is that it doesn't have a *critical* coupling: a non-trivial solution for the quark mass function and consequently also a chiral condensate exist for any $\Lambda > 0$. The gluon propagator in Eq. (2.143) is localized in momentum space because of the $\delta-$function. We could take the extreme opposite and localize it in coordinate space, which results in an effective four-fermi interaction between two quarks where the gluon shrinks to a point and is integrated out. This is the NJL model (Nambu, Jona-Lasinio), where the momentum dependence of the gluon is simply a constant:

$$D(k) \to \frac{1}{(2\pi)^2} \frac{c}{\Lambda^2} . \tag{2.149}$$

In this case it is more convenient to integrate over the quark momentum $q = p - k$ instead of k in (2.142); however, the self-energy integral must now be regulated because it is divergent. We could impose a sharp cutoff in q^2, so that the gluon propagator is a constant up to some scale Λ and zero above. As a consequence, the integral no longer contains the external momentum p and so $\Sigma(p)$ is also just a constant, which means that A and M will be constants as well. It turns out that the self-energy contribution to A vanishes and we get $A = 1$. The equation for M becomes:

$$M = m + cM \int_0^1 dy \, \frac{y}{y+a} =: m + cM \, f(a) , \qquad y = \frac{q^2}{\Lambda^2} , \quad a = \frac{M^2}{\Lambda^2} , \tag{2.150}$$

with $f(a) = 1 - a \ln(1 + \frac{1}{a})$. The function $f(a)$ satisfies $f(a) \leq 1$ and $f(0) = 1$. This means in the chiral limit we have again the trivial solution $M = 0$, but for $c \geq 1$ also a nontrivial solution is possible, where M as a function of c is determined from the equation $f(a) = 1/c$. The result is shown in Fig. 2.8. The 'quark mass' is zero for $c < 1$; above that value chiral symmetry is spontaneously broken. If we impose the same cutoff for the chiral condensate, we find that it is proportional to M as well, with a similar form as in Eq. (2.148): if we work out $M(c)$, the prefactor $2/15$ is replaced by a function of c which is zero as long as $c < 1$.

In general, the gluon propagator will be neither a $\delta-$function nor a constant, and the spontaneous breaking of chiral symmetry will not only generate a mass term for the quark propagator but also chirally asymmetric terms for the quark-gluon vertex. Nevertheless, both models encode general features:

- Implementing the scale Λ was necessary to make them work. If we replaced Λ^2 by k^2 in Eq. (2.143), the self-energy would vanish. In the NJL model, Λ defines the regularization cutoff which cannot be removed. Consequently, the mass function and other dimensionful quantities will be expressed in terms of that scale.

- Spontaneous chiral symmetry breaking is a *critical* phenomenon: if the combined strength from the gluon propagator and quark-gluon vertex (the 'effective' running coupling) exceeds a critical value, a quark mass will be generated dynamically; if this is not the case, we remain with the chirally symmetric solution.

- In contrast to some effective theories of QCD, where the terms that eventually lead to spontaneous symmetry breaking already appear in the Lagrangian, the original QCD Lagrangian tells us nothing about whether chiral symmetry is preserved or not at the quantum level. Its spontaneous breaking is a purely dynamical effect induced by the strong gluonic interactions, hence the name 'dynamical chiral symmetry breaking'.

Gell-Mann-Oakes-Renner relation. Now let's return to Goldstone's theorem. We have explored the origin of spontaneous chiral symmetry breaking and identified its order parameters: the scalar quark condensate or, equivalently, the quark mass function. Hence, any other quantity that depends on the mass function (and vanishes if the mass function does) will break chiral symmetry as well. In Eq. (2.79) we have found that as a simple consequence of the PCAC relation either a pseudoscalar meson's mass or its electroweak decay constant must be zero in the chiral limit. Therefore, if we can show that the pion decay constant is also generated by dynamical chiral symmetry breaking, we must have massless pions.

The right place to look for such a relation is the axial WTI in (2.56). On its right-hand side we have the sum of two quark propagators multiplied with γ_5; if we take the trace with another γ_5, it will become proportional to the quark condensate. If we shuffle the term with G_P to the left-hand side, we obtain a difference of AP and PP current correlators. When we insert the completeness relation, both of them will contain pseudoscalar poles *only*, and the residues will depend on f_λ. Moreover, the axial WTI tells us that all bound-state poles in G_A and G_P must cancel out with their numerators because the quark propagator does *not* exhibit such poles. To see this, let's start directly from the WTI for the current correlator from Eq. (2.50):

$$\partial_\mu^x \langle 0|\mathsf{T}\, A_a^\mu(x)\, P_b(0)|0\rangle - 2m\,\langle 0|\mathsf{T}\, P_a(x)\, P_b(0)|0\rangle = \delta(x^0)\,\langle 0|\left[A_a^0(x), P_b(0)\right]|0\rangle\,. \quad (2.151)$$

We already inserted the PCAC relation for the PP term. If we integrate over d^4x on the right-hand side, we obtain the vacuum expectation value of the commutator in Eq. (2.45),

$$\langle 0|\left[Q_a^A, P_b(0)\right]|0\rangle = -i\langle 0|\left[\frac{\delta_{ab}}{N_f}\,S(0) + d_{abc}\,S_c(0)\right]|0\rangle = -i\,\frac{\delta_{ab}}{N_f}\,\langle\bar\psi\psi\rangle\,, \quad (2.152)$$

where only the singlet condensate survives in the limit of exact $SU(N_f)_V$. This is the representative of the generic equation (2.131): since the condensate which is not invariant under axial symmetries is the scalar condensate and the respective charges are the axial charges, the corresponding field φ_i must be the pseudoscalar density. For the left-hand side in Eq. (2.151), we can insert the spectral decomposition from (2.82) and (2.85) and integrate over x. In momentum space, this means taking the limit $p \to 0$:

$$\lim_{p\to 0}\sum_\lambda \frac{p^2 f_\lambda - 2m\,r_\lambda}{p^2 - m_\lambda^2 + i\varepsilon}\,ir_\lambda\,\delta_{ab} = \sum_\lambda ir_\lambda f_\lambda\,\delta_{ab} = -i\,\frac{\delta_{ab}}{N_f}\,\langle\bar\psi\psi\rangle\,, \quad (2.153)$$

where we have used the relation $f_\lambda\, m_\lambda^2 = 2mr_\lambda$ from Eq. (2.79) in the second step. The poles cancel indeed, and we arrive at the result that if chiral symmetry is realized and the quark condensate vanishes, all combinations $r_\lambda\, f_\lambda$ must vanish as well; if it is spontaneously broken, there is at least one mode where both r_λ and f_λ are nonzero. Since $f_\lambda \neq 0$ in that case, we must have $m_\lambda \to 0$, i.e., a massless Goldstone boson. Each $|\lambda\rangle$ is proportional to one of the generators, so we have a massless Goldstone boson for each generator t_a (for three flavors with $SU(3)_A \times U(1)_A$ we obtain a pseudoscalar octet and a singlet). In turn, the decay constants f_λ must vanish for the remaining excited states with $m_\lambda \neq 0$, so we can remove the sum in the equation above and write

$$r_\lambda\, f_\lambda = -\frac{\langle\bar\psi\psi\rangle}{N_f}\,, \quad (2.154)$$

where λ is the ground state contribution. If we substitute r_λ by the condensate and insert it in Eq. (2.79), we obtain the Gell-Mann-Oakes-Renner (GMOR) relation,

$$f_\lambda^2\, m_\lambda^2 = -2m\,\frac{\langle\bar\psi\psi\rangle}{N_f}\,, \quad (2.155)$$

which is valid for each member of the lowest-lying pseudoscalar octet and singlet. (In the singlet case it only holds if we ignore the anomaly.)

In the derivation so far we have assumed that all quark masses are equal, $m_u = m_d = m_s$. In the case of $SU(3)_V$ breaking, we have to go back to the general PCAC relation (2.25) and evaluate the anticommutators, and also retain the d_{abc} terms in Eq. (2.152). In this case the GMOR relation retains its form for each generator with index a if we replace the quark mass m by

$$\begin{array}{ll} a=1,2,3: & \frac{1}{2}(m_u+m_d)\,, \\ a=4,5: & \frac{1}{2}(m_u+m_s)\,, \\ a=6,7: & \frac{1}{2}(m_d+m_s)\,, \end{array} \qquad \begin{array}{ll} a=8: & \frac{1}{6}(m_u+m_d+4m_s)\,, \\ a=0: & \frac{1}{3}(m_u+m_d+m_s)\,, \end{array} \tag{2.156}$$

and the condensate accordingly:

$$\frac{\langle\bar\psi\psi\rangle}{3} \longrightarrow \frac{\langle\bar u u+\bar d d\rangle}{2} \quad (a=1,2,3), \qquad \frac{\langle\bar u u+\bar s s\rangle}{2} \quad (a=4,5), \quad \text{etc.} \tag{2.157}$$

Therefore we get for the pions and kaons:

$$f_\pi^2\,m_\pi^2 = -\frac{m_u+m_d}{2}\,\langle\bar u u+\bar d d\rangle\,, \qquad f_K^2\,m_K^2 = -\frac{m_u+m_s}{2}\,\langle\bar u u+\bar s s\rangle\,. \tag{2.158}$$

Inserting the experimental values[12] $f_\pi \approx 92$ MeV, $m_\pi \approx 140$ MeV and assuming an average quark mass $m_u = m_d = 3.5$ MeV yields $\langle\bar u u\rangle = \langle\bar d d\rangle \approx -(280\,\text{MeV})^3$. The same estimate for kaons ($f_K \approx 110$ MeV, $m_K \approx 494$ MeV, $m_s \approx 120$ MeV) gives us $\langle\bar s s\rangle \approx -(290\,\text{MeV})^3$. The renormalized quark masses and condensates are renormalization-point and -scheme dependent; the values quoted here are consistent with recent lattice QCD results,[13] obtained in an $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV.

Strictly speaking, the GMOR relation as it stands is only valid in the chiral limit because the quark condensate is only well-defined for $m = 0$. We can see this from its definition (2.141) as the momentum integral of the quark mass function: if $M(p^2 \to \infty)$ doesn't vanish like $1/p^2$ but rather logarithmically (which happens for $m \neq 0$), the integral diverges quadratically. In this case, $f_\lambda\, m_\lambda^2 = 2mr_\lambda$ from Eq. (2.79) can be viewed as the generalized GMOR relation since the quantities f_λ and r_λ are well-defined for all quark masses. In principle, they can be used to define the scalar quark condensate entirely from a pseudoscalar meson's bound-state wave function, namely as the chiral limit of the combination $f_\lambda\, r_\lambda$ via Eq. (2.154).

[12]The decay constants are often defined with a factor $\sqrt{2}$, so that $f_\pi \approx 130$ MeV.
[13]McNeile et al., Phys. Rev. D87 (2013), 034503. arXiv:1211.6577.