2.5 $U(1)_A$ anomaly

We have seen that spontaneous chiral symmetry breaking should affect all axial symmetries including the flavor-singlet $U(1)_A$. The fact that there is no good candidate for a flavor-singlet (pseudo-) Goldstone boson in the spectrum is related to the anomalous $U(1)_A$ breaking. Anomalies are symmetries of classical Lagrangians that are broken at the quantum level. They arise when regularization destroys a symmetry and there is no regulator choice that can preserve it. Since the symmetry is lost, there is no Goldstone boson because the quantum corrections will generate a mass for that mode.

Anomalies are again a typical feature of axial symmetries. In contrast to spontaneous symmetry breaking, where the symmetry is lost due to dynamical effects, anomalies have their origin in short-distance singularities of the currents $A^\mu_a = \overline{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi$ and $A^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\psi$. These are composite operators at the same space-time point which are potentially divergent and have to be regularized. In principle, the problem would also affect vector currents, but in that case it is possible to find appropriate regularization prescriptions that leave their symmetry intact. Vector symmetries are related to conserved charges (color charge, electromagnetic charge, flavor charges, etc.). If they were broken at the quantum level, we would not only lose charge conservation but also gauge symmetry, and the theory would become nonrenormalizable and inconsistent. Global axial symmetries are in that sense 'less important' and the fact that they produce anomalies is not a serious problem for the theory.[14] In the following we will see that QCD leads only to an anomalous $U(1)_A$ breaking, whereas QED also induces an anomalous $SU(N_f)_A$ breaking.

Anomalies from the path integral. We have anticipated in Eq. (2.29) that the divergence of the axialvector singlet current picks up an anomalous contribution

$$\partial_\mu A^\mu = 2i\,\overline{\psi}\,\mathsf{M}\,\gamma_5\,\psi + N_f\,\mathcal{Q}(x)\,, \tag{2.159}$$

where $\mathcal{Q}(x)$ is the topological charge density that we encountered in Section 1.1:

$$\mathcal{Q}(x) := \frac{g^2}{8\pi^2}\,\mathrm{Tr}\,\{\widetilde{F}_{\mu\nu}\,F^{\mu\nu}\,\}, \qquad \widetilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,. \tag{2.160}$$

The derived relation (2.108) entails that the mass of the η_0 does not vanish in the chiral limit, so there is no flavor-singlet Goldstone boson. To see how this term comes about, start with the massless Lagrangian $\mathcal{L} = \overline{\psi}\,i\not{D}\,\psi$ and perform an axial $U(1)_A$ rotation with a group parameter $\varepsilon(x)$:

$$\mathcal{L}' = \overline{\psi}\,e^{i\varepsilon\gamma_5}\,i\not{D}\,e^{i\varepsilon\gamma_5}\,\psi = \overline{\psi}\,(i\not{D} - \not{\partial}\,\varepsilon\,\gamma_5)\,\psi = \mathcal{L} - (\partial_\mu\varepsilon)\,A^\mu\,. \tag{2.161}$$

If we put this in the path integral, we have

$$Z[0] = \int \mathcal{D}[\psi',\overline{\psi}']\,e^{iS[\psi',\overline{\psi}']} = \int \mathcal{D}[\psi,\overline{\psi}]\,e^{iS[\psi,\overline{\psi}]}\left(1 + i\int d^4x\,\varepsilon(x)\,\partial_\mu A^\mu(x)\right), \tag{2.162}$$

[14] Except when they are also promoted to gauge symmetries: if a gauge symmetry is broken anomalously, then one needs anomaly cancellations between different sectors of the theory.

and hence $\langle \partial_\mu A^\mu \rangle = 0$, i.e., current conservation holds inside the expectation value.[15] As always we have assumed that the path integral measure remains invariant under the transformation. However, for axial transformations this is not necessarily the case. The origin of this behavior is the transformation of the Dirac spinors

$$\psi'(x) = e^{+i\varepsilon\gamma_5}\psi\,, \qquad \overline{\psi}'(x) = \overline{\psi}(x)\, e^{+i\varepsilon\gamma_5}\,, \tag{2.163}$$

which leads to a Jacobian determinant of the transformation:

$$\mathcal{D}[\psi', \overline{\psi}'] = (\det C)^{-2}\, \mathcal{D}[\psi, \overline{\psi}]\,. \tag{2.164}$$

It turns out that this determinant is ill-defined and requires regularization, which in turn breaks the $U(1)_A$ symmetry. The final result is exactly the anomalous term:

$$(\det C)^{-2} = \exp\left(-i\int d^4x\, \varepsilon(x)\, N_f\, \mathcal{Q}(x)\right). \tag{2.165}$$

Putting this back in Eq. (2.162) yields the anomalous correction to the PCAC relation:

$$Z[0] = \int \mathcal{D}[\psi, \overline{\psi}]\, e^{iS[\psi,\overline{\psi}]}\left(1 + i\int d^4x\, \varepsilon(x)\, \big(\underbrace{\partial_\mu A^\mu(x) - N_f\, \mathcal{Q}(x)}_{\stackrel{!}{=}0}\big)\right). \tag{2.166}$$

Fujikawa's method. In order to calculate this, we can expand the functional determinant into eigenfunctions of the Dirac operator. The hermitian Dirac operator $\not{D}$ has real eigenvalues λ_n and a set of orthonormal, complete eigenfunctions:[16]

$$\not{D}\,\varphi_n(x) = \lambda_n\,\varphi_n(x)\,, \qquad \begin{aligned} \int d^4x\, \varphi^\dagger_{m,i}(x)\, \varphi_{n,j}(x) &= \delta_{mn}\,\delta_{ij}\,, \\ \textstyle\sum_n \varphi_{n,i}(x)\, \varphi^\dagger_{n,j}(y) &= \delta^4(x-y)\,\delta_{ij}\,, \end{aligned} \tag{2.167}$$

where i, j collect the remaining Dirac, color and flavor indices. Then we can expand the spinors ψ, $\overline{\psi}$ into these eigenfunctions, where the coefficients a_n and $\bar{b}_n$ are independent Grassmann variables, and write down the path integral measure:

$$\psi(x) = \sum_n a_n\,\varphi_n(x)\,, \quad \overline{\psi}(x) = \sum_n \varphi^\dagger_n(x)\,\bar{b}_n\,, \quad \mathcal{D}[\psi,\overline{\psi}] = \prod_n da_n \prod_m d\bar{b}_m\,. \tag{2.168}$$

The fermionic path integral can be written as the determinant of the Dirac operator:

$$\det \not{D} = \int \mathcal{D}[\psi,\overline{\psi}]\, e^{i\int d^4x\,\overline{\psi}\, i\not{D}\,\psi} = \int \prod_n da_n\, d\bar{b}_n\, e^{-\sum_n \bar{b}_n\,\lambda_n\, a_n} = \prod_n \lambda_n\,. \tag{2.169}$$

[15]The proper argument would start from the background field method in Section 2.1: add a source term $B \cdot A$ with $\delta B_\mu = \partial_\mu \varepsilon$ to the Lagrangian in the path integral exponential, with the purpose to cancel the additional term in (2.162) from the local $U(1)_A$ gauge transformation. The resulting partition function is locally gauge invariant, and one derives the WTI as in Eq. (2.64).

[16]To ensure (anti-)hermiticity of the Dirac operator, we should actually do this in Euclidean space.

An axial transformation changes the coefficients a_n and $\bar{b}_n$ to

$$a_n' = \int d^4x\, \varphi_n^\dagger(x)\, \psi'(x) = \sum_m \underbrace{\int d^4x\, \varphi_n^\dagger(x)\, e^{i\varepsilon(x)\gamma_5}\, \varphi_m(x)}_{=:C_{nm}}\, a_m \tag{2.170}$$

so that we have

$$a_n' = \sum_m C_{nm}\, a_m\,, \qquad \bar{b}_m' = \sum_n C_{nm}\, \bar{b}_n\,. \tag{2.171}$$

Because the Grassmann measure transforms with the inverse determinant we arrive at Eq. (2.164). Using $\det C = e^{\mathrm{Tr}\,\ln C}$ and expanding the logarithm, we obtain

$$(\det C)^{-2} = \exp\left(-2i \int d^4x\, \varepsilon(x) \sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x)\right), \tag{2.172}$$

which involves the 'functional trace' over γ_5. With the completeness relation in (2.167), the sum becomes

$$\sum_n \varphi_n^\dagger(x)\, \gamma_5\, \varphi_n(x) = \lim_{y\to x} \mathrm{Tr}\,\{\gamma_5\}\, \delta^4(x-y)\,, \tag{2.173}$$

where the trace is now the usual trace over Dirac, color and flavor indices. (The color-flavor trace would just produce a factor $N_f\, N_C$.) Normally, this trace would be zero and the determinant 1, but because of the short-distance singularity the expression is not well-defined $(0\cdot\infty)$ and must be regulated.

Fujikawa suggested to damp the contribution from the large eigenvalues by a Gaussian cutoff with regulator mass M which is taken to infinity in the end:

$$\begin{aligned} \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\, \gamma_5\, e^{-(\lambda_n/M)^2}\, \varphi_n(x) &= \lim_{M\to\infty} \sum_n \varphi_n^\dagger(x)\, \gamma_5\, e^{-(\not{D}/M)^2}\, \varphi_n(x) \\ &= \lim_{\substack{M\to\infty \\ y\to x}} \mathrm{Tr}\left\{\gamma_5\, e^{-(\not{D}/M)^2}\right\} \delta^4(x-y). \end{aligned} \tag{2.174}$$

The regularization is gauge-invariant because the covariant derivative appears in it; hence, it preserves the vector symmetry. To proceed, one can express $\not{D}^2$ as

$$\not{D}^2 = \gamma^\mu\gamma^\nu D_\mu D_\nu = \frac{1}{2}\{\gamma_\mu,\gamma_\nu\}\, D^\mu D^\nu + \frac{1}{2}[\gamma_\mu,\gamma_\nu]\, D^\mu D^\nu = D^2 - \frac{ig}{4}[\gamma_\mu,\gamma_\nu]\, F^{\mu\nu}. \tag{2.175}$$

With the generic relation $f(\partial_x)\, e^{ikx} = e^{ikx} f(\partial_x + ik)$, where unsaturated derivatives in the final expression vanish, we can take Eq. (2.174) to momentum space:

$$\lim_{y\to x} f(\partial_x)\, \delta^4(x-y) = \lim_{y\to x} f(\partial_x) \int \frac{d^4k}{(2\pi)^4}\, e^{ik\,(x-y)} = \int \frac{d^4k}{(2\pi)^4}\, f(\partial_x + ik)\,, \tag{2.176}$$

so one has to expand the exponential in (2.174) and replace $\partial_\mu \to \partial_\mu + ik_\mu$. The lowest nonvanishing contribution is the square of the commutator term $\sim F^{\mu\nu}$ in Eq. (2.175) because the trace with γ_5 requires four $\gamma-$matrices to be nonzero:

$$\frac{i}{4}\, \mathrm{Tr}\left\{\gamma_5\, \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right\} = \varepsilon^{\mu\nu\alpha\beta}\,. \tag{2.177}$$

The resulting expansion has the form

$$\text{Eq. (2.174)} = \lim_{M\to\infty}\left[\frac{1}{M^4}\int\frac{d^4k}{(2\pi)^4}\,e^{-\frac{k^2}{M^2}}\,g^2 N_f\,\text{Tr}\,\{\widetilde{F}_{\mu\nu}\,F^{\mu\nu}\} + \frac{1}{M^6}\,(\dots) + \dots\right]. \tag{2.178}$$

After sending $M\to\infty$ and integrating out the momentum k, one obtains

$$\sum_n \varphi_n^\dagger(x)\,\gamma_5\,\varphi_n(x) = \lim_{y\to x}\,\text{Tr}\,\{\gamma_5\}\,\delta^4(x-y) = \frac{g^2 N_f}{16\pi^2}\,\text{Tr}\,\{\widetilde{F}_{\mu\nu}\,F^{\mu\nu}\}, \tag{2.179}$$

where the trace over the $SU(3)_C$ generators in $F^{\mu\nu} = F_a^{\mu\nu}\,\mathsf{t}_a$ remains. Inserted in the determinant (2.172), we arrive at the result in Eq. (2.165).

Remarks:

- Note that we did not perform an 'additional renormalization' because the theory was already renormalized before. Renormalization means that the regulator remains in the theory, but it is hidden in the renormalization constants which must cancel each other in observables. Here we have merely cured a $0\cdot\infty$ situation by introducing a cutoff M that we sent to infinity at the end; however, the resulting finite expression has the property that it breaks the $U(1)_A$ symmetry. While we used exponential damping, one can show that this result is indeed independent of the chosen regularization as long as it is gauge invariant.

- Since the topological charge is essentially the trace over γ_5, one can ask why the non-Abelian global $SU(N_f)_A$ transformations do not lead to anomalies. Repeating the analysis with $\varepsilon\to\sum_a \varepsilon_a\,\mathsf{t}_a$ yields:

$$\partial_\mu A_a^\mu = \frac{g^2}{(4\pi)^2}\,\varepsilon^{\alpha\beta\mu\nu}\,F^b_{\alpha\beta}\,F^c_{\mu\nu}\,\text{Tr}_F\,\{\mathsf{t}_a\}\,\text{Tr}_C\,\{\mathsf{t}_b\,\mathsf{t}_c\}\,, \tag{2.180}$$

  which vanishes in the flavor-octet case because $\text{Tr}\{\mathsf{t}_a\}=0$. In other words, gluons couple only to flavor-singlet currents, and the anomaly signals the breakdown of the $U(1)_A$ symmetry in the presence of gluons.

- The topological charge density can be written as the divergence of a current, the Chern-Simons current:

$$\mathcal{Q}(x) = \partial_\mu K^\mu\,, \qquad K^\mu = \frac{g^2}{8\pi^2}\,\varepsilon^{\mu\nu\alpha\beta}\,\text{Tr}\left\{F_{\alpha\beta}\,A_\nu + \frac{2ig}{3}A_\alpha A_\beta A_\nu\right\}. \tag{2.181}$$

  One could then conclude that the flavor-singlet PCAC relation (in the chiral limit) still induces a conserved current, which leads back to the argument that there should be a flavor-singlet Goldstone boson. However, K^μ and its corresponding charge $\int d^3x\,K^0$ are not gauge invariant, so they cannot couple to physical states and hence there is no conserved axial charge.

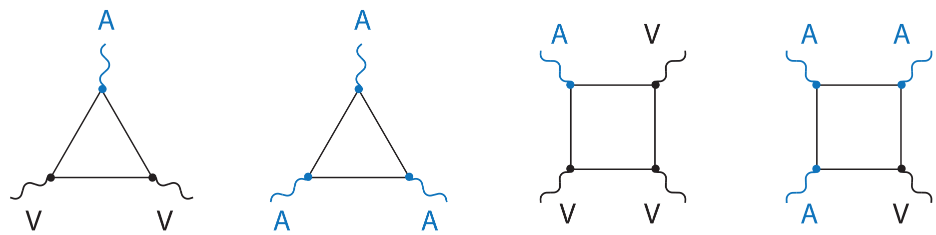

FIGURE 2.9: Anomalous 1-loop fermion diagrams.

Triangle diagrams. The axial anomaly will show up (and was originally derived) in the calculation of Green functions that involve axialvector currents, e.g.

$$\langle 0|\mathsf{T}\, A^\mu(x)\, V^\alpha(y)\, V^\beta(z)\,|0\rangle\,, \quad \langle 0|\mathsf{T}\, A^\mu(x)\, A^\alpha(y)\, A^\beta(z)\,|0\rangle\,, \quad \text{etc.} \tag{2.182}$$

Take for example the axialvector and vector WTIs for an AVV correlator:

$$\begin{aligned}\partial^x_\mu \,\langle A^\mu V^\alpha V^\beta\rangle = \langle\, (\partial_\mu A^\mu) V^\alpha V^\beta\rangle + \delta(x^0-y^0)\,\langle [A^0, V^\alpha]\, V^\beta\rangle \\ + \delta(x^0-z^0)\,\langle V^\alpha\,[A^0, V^\beta]\rangle = 0\,.\end{aligned} \tag{2.183}$$

The last two terms on the right-hand side are zero because the commutators of the singlet currents vanish, cf. (2.42). The first term produces the pseudoscalar density via the PCAC relation. Repeating this for derivatives with respect to y and z, we arrive at

$$\partial^x_\mu\,\langle A^\mu V^\alpha V^\beta\rangle = 2m\,\langle P V^\alpha V^\beta\rangle\,, \quad \partial^y_\alpha\,\langle A^\mu V^\alpha V^\beta\rangle = 0, \quad \partial^z_\beta\,\langle A^\mu V^\alpha V^\beta\rangle = 0, \tag{2.184}$$

without taking into account the anomaly. These diagrams are linearly divergent and therefore not translationally invariant. If one calculates them explicitly to 1-loop order, shifting integration variables by a different momentum routing will produce results that differ by surface terms. The freedom in distributing these surface terms can be used in the regularization procedure when getting rid of all infinite pieces. It turns out that the relations (2.184) cannot be satisfied simultaneously, and in order to preserve the vector symmetries the axialvector WTI must pick up the additional anomalous term.

A theorem by Adler and Bardeen states that the full structure of the anomaly is already contained in the perturbative one-loop fermion diagrams. Higher-loop corrections do not renormalize the anomaly except for replacing the fields and coupling constants by their renormalized values. For anomaly considerations it is therefore enough to calculate the triangle and rectangle diagrams in Fig. 2.9. These are the superficially divergent ones (in fact, pentagon diagrams should be included as well although they are convergent), and they include an odd number of axial currents and thus an odd number of γ_5 matrices.

QED anomaly and $\pi^0 \to \gamma\gamma$ decay. Anomalies have observable consequences. The prime example are the η and η' masses, but in that case the anomalous contribution is also difficult to quantify due to the explicit breaking of chiral symmetry and mixing effects. A much cleaner system is the decay of the π^0 into two photons, which is almost exclusively caused by the axial anomaly from QED effects.

So far we have considered the axial anomaly in QCD (the 'gluon anomaly') which is the relevant one for the $\eta - \eta'$ problem. Quarks couple to gluons, and the quark's flavor-singlet axialvector current A^μ picks up an anomalous term containing the gluonic field-strength tensor. On the other hand, quarks can also couple to photons which will also produce an anomaly, although the related effects are much weaker ($\alpha_{\text{QED}} \ll \alpha_{\text{QCD}}$). If we repeat the derivation for the QED Lagrangian, replace $F^{\mu\nu}$ by the electromagnetic field-strength tensor and the coupling g with e, we obtain the electromagnetic 'photon anomaly' (Adler-Bell-Jackiw or ABJ anomaly):

$$\partial_\mu A^\mu_a = \frac{e^2}{(4\pi)^2}\,\varepsilon^{\alpha\beta\mu\nu}\,F_{\alpha\beta}\,F_{\mu\nu}\,\text{Tr}_\text{F}\left\{\mathsf{t}_a \mathsf{Q}^2\right\}\text{Tr}_\text{C}\left\{1\right\}, \qquad \mathsf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.185}$$

stated here without the fermion mass term. Since fermions with different flavors have different charges (expressed by the quark charge matrix Q), photons can also couple to flavor-nonsinglet currents. Therefore, the electromagnetic anomaly produces additional terms for the divergences of the axial currents A^μ and A^μ_a.

To extract the $\pi^0 \to \gamma\gamma$ decay, consider the three-point function of an axialvector current and two electromagnetic vector currents:

$$\langle 0|\mathsf{T}\, A^\mu_a(x)\, V^\alpha_{\text{em}}(x_1)\, V^\beta_{\text{em}}(x_2)|0\rangle\,. \tag{2.186}$$

The electromagnetic current is proportional to the quark charges and thus given by

$$V^\mu_{\text{em}}(x) = \bar\psi(x)\,\gamma^\mu\,\mathsf{Q}\,\psi(x) = V^\mu_3(x) + \tfrac{1}{\sqrt{3}}\,V^\mu_8(x)\,. \tag{2.187}$$

To lowest order perturbation theory, Eq. (2.186) is the AVV triangle diagram of Fig. 2.9 and diverges linearly. However, it has also a nonperturbative spectral representation in terms of pseudoscalar bound-state poles. We can derive this in complete analogy to Eqs. (2.151)–(2.153). First, write down its Ward-Takahashi identity by acting with the derivative on the index μ:

$$\partial^x_\mu\,\langle 0|\mathsf{T}\,A^\mu_a(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle - 2m\,\langle 0|\mathsf{T}\,P_a(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle = \dots \tag{2.188}$$

We are interested in the π^0 with $a = 3$; in that case the commutators on the right-hand side obtained from (2.52) vanish, because they contain the structure constants $f_{338} = 0$, etc. Instead we have the contribution from the anomaly:

$$\dots = \frac{e^2 D}{(4\pi)^2}\,\varepsilon^{\alpha\beta\rho\sigma}\,\langle 0|\mathsf{T}\,F_{\alpha\beta}(x)\,F_{\rho\sigma}(x)\,V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\,V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle\,, \tag{2.189}$$

where the factor $D = N_C/6$ comes from the flavor and color traces.

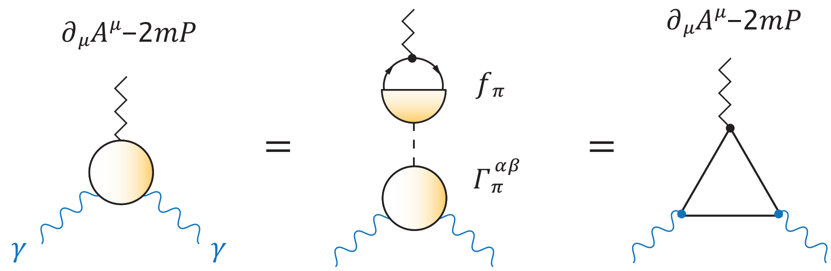

FIGURE 2.10: $\pi^0 \to \gamma\gamma$ decay in the chiral limit.

If we work out the time orderings on the left-hand side and insert the completeness relation, we can again isolate the Feynman propagator. The pole residues are the two decay constants from Eq. (2.78) and the $\pi \to \gamma\gamma$ decay amplitude, defined via

$$i\langle\lambda|\mathsf{T}\, V^\alpha_{\text{em}}\big(\tfrac{z}{2}\big)\, V^\beta_{\text{em}}\big(-\tfrac{z}{2}\big)|0\rangle =: \Gamma^{\alpha\beta}_\lambda(z,p) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iqz}\, \Gamma_\lambda(q,p)\, \varepsilon^{\alpha\beta\rho\sigma} q^\rho p^\sigma\,. \tag{2.190}$$

Its expression in momentum space is due to Lorentz and parity invariance: p is the pion momentum, q is the relative momentum between the photons, and the only negative-parity Lorentz tensor that depends on two momenta is the one above. If we integrate (2.188) over x and z, the poles drop out again and the analogue of Eq. (2.153) becomes

$$\lim_{\substack{p\to 0\\ q\to 0}} \sum_\lambda f_\lambda\, \Gamma^{\alpha\beta}_\lambda(q,p) = \lim_{\substack{p\to 0\\ q\to 0}} f_\pi\, \Gamma^{\alpha\beta}_\pi(q,p) = 0\,, \tag{2.191}$$

as long as we discard the anomaly on the right-hand side. We have again removed the sum over λ because the decay constants are zero for all excited states with $m_\lambda \neq 0$. Since the transition matrix elements are defined at $p^2 = m_\pi^2 = 0$, this is a chiral-limit relation. Hence, the decay amplitude should be zero, which is known as the Sutherland-Veltman theorem.

In order to take the anomaly into account, we would have to work out the right-hand side of Eq. (2.189). However, since the anomaly is produced already in the lowest order perturbation theory, it is sufficient to start again from Eq. (2.188) and work out its perturbative 1-loop contributions, the AVV and PVV triangle diagrams. The ambiguity in shifting integration variables produces just the anomalous term.[17] The result has the same structure in momentum space $\sim \varepsilon^{\alpha\beta\rho\sigma} q^\rho p^\sigma$, and the transition form factor in Eq. (2.191) becomes

$$\Gamma_\pi(0,0) = \frac{e^2 D}{2\pi^2 f_\pi}\,. \tag{2.192}$$

The calculated $\pi \to \gamma\gamma$ decay width using this result is 7.862 eV; the experimental value is 7.8 ± 0.9 eV. Therefore, the neutral pion decay doesn't probe the nonperturbative structure of QCD at all — it is practically completely determined by the axial anomaly.

[17] The calculation can be found in most textbooks, for example Kaku, p.414ff.