5 Electromagnetic field

Classical electromagnetism. Classical Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho\,,\quad \boldsymbol{\nabla}\times\boldsymbol{B}-\frac{\partial\boldsymbol{E}}{\partial t}=\boldsymbol{j}\,,\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0\,,\quad \boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0\,. \tag{5.1}$$

The inhomogeneous equations imply local charge conservation:

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0\,. \tag{5.2}$$

To arrive at covariant equations, define the current $j^\mu=(\rho,\boldsymbol{j})$ and the antisymmetric field-strength tensor $F^{\mu\nu}=-F^{\nu\mu}$ as

$$F^{ij}=-\varepsilon_{ijk}\,B^k\quad\Leftrightarrow\quad B^i=-\tfrac{1}{2}\,\varepsilon_{ijk}\,F^{jk}\,,\qquad F^{0i}=-E^i\,, \tag{5.3}$$

together with its dual:

$$\widetilde{F}^{\mu\nu}=\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma}\qquad\Rightarrow\qquad \begin{array}{ll}\widetilde{F}^{ij} &= \varepsilon_{ijk}\,F^{0k}=-\varepsilon_{ijk}\,E^k\,,\\ \widetilde{F}^{0i} &= -\tfrac{1}{2}\,\varepsilon_{ijk}\,F^{jk}=B^i\,.\end{array} \tag{5.4}$$

The combination of Maxwell equations and current conservation becomes

$$\partial_\mu\,F^{\mu\nu}=j^\nu\,,\qquad \partial_\mu\,\widetilde{F}^{\mu\nu}=0\,,\qquad \partial_\mu\,j^\mu=0\,. \tag{5.5}$$

Current conservation follows again from the inhomogeneous Maxwell equation because $\partial_\mu\partial_\nu F^{\mu\nu}=0$. The homogeneous Maxwell equations allow us to construct a vector potential $A^\mu=(\phi,\boldsymbol{A})$ via

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\qquad\Leftrightarrow\qquad \boldsymbol{E}=-\boldsymbol{\nabla}\phi-\frac{\partial\boldsymbol{A}}{\partial t}\,,\quad \boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}\,, \tag{5.6}$$

which is then only determined up to a derivative:

$$A'^\mu(x)=A^\mu(x)+\partial^\mu\varepsilon(x)\qquad\Leftrightarrow\qquad \phi'=\phi+\frac{\partial\varepsilon}{\partial t}\,,\quad \boldsymbol{A}'=\boldsymbol{A}-\boldsymbol{\nabla}\varepsilon\,. \tag{5.7}$$

In other words, $F^{\mu\nu}$ and therefore the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant under **local gauge transformations**, and vector fields A^μ that differ only by such a term are physically equivalent. Local gauge invariance will eventually become the fundamental construction principle for interacting field theories. At the present stage it merely corresponds to a redundancy in the description of the system, and to determine the true physical degrees of freedom we must be sure to divide out this redundancy (which will be the main difficulty in quantizing the system). In summary, all three equations in Eq. (5.5) can be combined into the Maxwell equations

$$\Box\,A^\mu-\partial^\mu\partial_\nu A^\nu=j^\mu\,. \tag{5.8}$$

Lagrangian of the electromagnetic field. We interpret the vector field $A^\mu(x)$ now as the fundamental electromagnetic field. The Maxwell equations follow as the equations of motion from the action

$$S = \int d^4x\,\mathcal{L} = \int d^4x \left[-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \right] = \int d^4x \left[\frac{1}{2}\,(\boldsymbol{E}^2 - \boldsymbol{B}^2) - j^\mu A_\mu \right]. \tag{5.9}$$

The current $j^\mu(x)$ that appears here as a static source term is presently just a compromise that we will eventually get rid of: in a fully interacting theory the current will emerge from other fields and thereby carry their dynamical information (in a free field theory $j^\mu = 0$). Let's rewrite the action in terms of A^μ and its derivatives:

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{2}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,\partial^\mu A^\nu - j^\mu A_\mu \right] \\ &\stackrel{\text{p.I.}}{=} \int d^4x \left[\frac{1}{2}\,A_\mu\,(\Box\, g^{\mu\nu} - \partial^\mu\partial^\nu)\,A_\nu - j^\mu A_\mu \right]. \end{aligned} \tag{5.10}$$

From the first line above it is easy to derive the Maxwell equations via

$$\frac{\partial\mathcal{L}}{\partial A^\nu} = -j_\nu\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = -\partial^\mu A_\nu + \partial_\nu A^\mu = -F^\mu{}_\nu\,, \tag{5.11}$$

and we can read off the canonical conjugate momentum:

$$\Pi_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\nu)} = -F^0{}_\nu \qquad \Rightarrow \qquad \Pi^0 = 0\,, \quad \boldsymbol{\Pi} = \boldsymbol{E}\,. \tag{5.12}$$

Note that the time component A_0 has no conjugate momentum, which will produce difficulties in the quantization. The Hamilton function becomes

$$\begin{aligned} H = \int d^3x\,\mathcal{H} &= \int d^3x \left[\Pi_\nu \frac{\partial A^\nu}{\partial t} - \mathcal{L} \right] \\ &= \int d^3x \left[\frac{1}{2}\,(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \boldsymbol{E}\cdot\boldsymbol{\nabla}\phi + \rho\,\phi - \boldsymbol{j}\cdot\boldsymbol{A} \right] \\ &\stackrel{\text{p.I.}}{=} \int d^3x \left[\frac{1}{2}\,(\boldsymbol{E}^2 + \boldsymbol{B}^2) - \boldsymbol{j}\cdot\boldsymbol{A} \right]. \end{aligned} \tag{5.13}$$

Poincaré transformations. Let's study the conservation laws that follow from the Poincaré invariance of the action. According to Eq. (1.40), the generic infinitesimal current takes the form

$$-\,\delta j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\,\delta A^\nu - T^{\mu\nu}\delta x_\nu = -F^{\mu\nu}\,\delta A_\nu - T^{\mu\nu}\delta x_\nu\,, \tag{5.14}$$

and the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\alpha)}\,\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L} = -F^{\mu\alpha}\,\partial_\nu A^\alpha - g^{\mu\nu}\mathcal{L}\,. \tag{5.15}$$

Translation invariance ($\delta A^\nu = 0$, $\delta x_\nu = a_\nu$) implies that it is conserved; however, in the presence of the current $j^\mu(x)$ its divergence is $\partial_\mu T^{\mu\nu} = (\partial^\nu j_\alpha)A^\alpha \neq 0$. This is just because we treat the current as an external source: in a complete theory it would emerge from other fields which also contribute to the energy-momentum tensor. (By the way, note that δj^μ has nothing to do with j^μ.)

Under Lorentz transformations the field transforms as

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu \qquad \Leftrightarrow \qquad \begin{aligned} \delta x_\alpha &= \varepsilon_{\alpha\beta}\, x^\beta\,, \\ \delta A_\alpha &= \varepsilon_{\alpha\beta} A^\beta = \tfrac{i}{2}\, \varepsilon_{\mu\nu}\, (M^{\mu\nu})_{\alpha\beta}\, A^\beta\,, \end{aligned} \tag{5.16}$$

because in the vector representation the irreducible representation matrix is just the Lorentz transformation itself. The infinitesimal generator of Lorentz transformations was given in Eq. (2.54):

$$(M^{\mu\nu})_{\alpha\beta} = -i\,(\delta^\mu_\alpha\,\delta^\nu_\beta - \delta^\nu_\alpha\,\delta^\mu_\beta)\,. \tag{5.17}$$

The corresponding infinitesimal current defines the angular momentum density,

$$-\,\delta j^\mu = -\tfrac{1}{2}\,\varepsilon_{\alpha\beta}\,\underbrace{\left(F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha + T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha\right)}_{=:\, m^{\mu,\alpha\beta}}\,, \tag{5.18}$$

which is conserved (if the external current $j^\mu = 0$): $\partial_\mu\, m^{\mu,\alpha\beta} = 0$. Once again we can insert the explicit form of the energy-momentum tensor and isolate the orbital angular-momentum part:

$$T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha = iF^{\mu\rho}\, L^{\alpha\beta} A_\rho + (x^\alpha g^{\mu\beta} - x^\beta g^{\mu\alpha})\mathcal{L}\,, \tag{5.19}$$

where $L^{\alpha\beta}$ was defined in Eq. (1.44). In combination with the spin contribution, the angular momentum density becomes

$$m^{\mu,\alpha\beta} = iF^{\mu\rho}\big[(M^{\alpha\beta})_{\rho\sigma} + g_{\rho\sigma} L^{\alpha\beta}\big] A^\sigma =: iF^{\mu\rho}\,(J^{\alpha\beta})_{\rho\sigma} A^\sigma\,. \tag{5.20}$$

Hence, the charge that is conserved under rotations is the angular momentum of the electromagnetic field

$$\int d^3x\, m^{0,ij} = -i \int d^3x\, E^k (J^{ij})_{kl} A^l =: -\varepsilon_{ijk}\, \widetilde{J}^k\,, \tag{5.21}$$

whose explicit form is

$$\widetilde{\boldsymbol{J}} = \int d^3x\, \Big(\boldsymbol{E} \times \boldsymbol{A} - iE^k \boldsymbol{L}\, A^k\Big) \tag{5.22}$$

with $\boldsymbol{L} = \boldsymbol{x} \times (-i\boldsymbol{\nabla})$. The spin of the electromagnetic field is $\int d^3x\, \boldsymbol{E} \times \boldsymbol{A}$.

(Ex) The energy-momentum tensor $T^{\mu\nu}$ in Eq. (5.15) is neither symmetric in its indices nor gauge-invariant (because it depends explicitly on A^α). An alternative *symmetric* form of the energy-momentum tensor is the **Belinfante tensor**, which is still conserved and therefore physically equivalent:

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - \tfrac{1}{2}\,\partial_\mu\,(s^{\mu,\alpha\beta} + s^{\alpha,\beta\mu} - s^{\beta,\mu\alpha})\,. \tag{5.23}$$

Here, $s^{\mu,\alpha\beta}$ is the spin contribution to the angular momentum density, i.e.

$$m^{\mu,\alpha\beta} = s^{\mu,\alpha\beta} + T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha\,. \tag{5.24}$$

This statement is general and holds independently of the nature of the fields. Its proof is simple: by construction, $s^{\mu,\alpha\beta}$ is antisymmetric in α, β and therefore

$$\Theta^{\alpha\beta} - \Theta^{\beta\alpha} = T^{\alpha\beta} - T^{\beta\alpha} - \partial_\mu\, s^{\mu,\alpha\beta}\,. \tag{5.25}$$

On the other hand, by taking the derivative of $m^{\mu,\alpha\beta}$ we see that

$$\partial_\mu\, m^{\mu,\alpha\beta} = \partial_\mu\, s^{\mu,\alpha\beta} - (T^{\alpha\beta} - T^{\beta\alpha}) = 0 \tag{5.26}$$

and therefore $\Theta^{\alpha\beta}$ is symmetric. (We used the fact that $T^{\alpha\beta}$ and $m^{\mu,\alpha\beta}$ are conserved.) $\Theta^{\alpha\beta}$ is conserved because the bracket in Eq. (5.23) is antisymmetric under an exchange $\mu \leftrightarrow \alpha$:

$$\partial_\alpha\, \Theta^{\alpha\beta} = -\tfrac{1}{2}\, \partial_\alpha \partial_\mu\, (s^{\mu,\alpha\beta} - s^{\alpha,\mu\beta} - s^{\beta,\mu\alpha}) = 0\,. \tag{5.27}$$

Let's work out the Belinfante tensor for the electromagnetic field. When inserting $s^{\mu,\alpha\beta} = F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha$ into Eq. (5.23) we obtain

$$\begin{aligned} \Theta^{\alpha\beta} &= T^{\alpha\beta} - \partial_\mu\, (F^{\mu\alpha} A^\beta) = -F^{\alpha\mu}\, \partial^\beta A_\mu - \partial_\mu\, (F^{\mu\alpha} A^\beta) - g^{\alpha\beta} \mathcal{L} \\ &= F^{\alpha\mu} F_\mu{}^\beta - j^\alpha A^\beta - g^{\alpha\beta} \mathcal{L}\,. \end{aligned} \tag{5.28}$$

Apart from the $j \cdot A$ term it is now also gauge-invariant. (Although $T^{\mu\nu}$ was gauge dependent, the charges derived from it are gauge-invariant because gauge transformations would only produce surface terms – see Maggiore, p.68.) Its components are

$$\begin{aligned} \Theta^{00} &= (F^{0i})^2 - j^0 A^0 - \mathcal{L} = \frac{1}{2}\, (\boldsymbol{E}^2 + \boldsymbol{B}^2) - \boldsymbol{j} \cdot \boldsymbol{A} = \mathcal{H}\,, \\ \Theta^{0i} &= F^{0k}\, F^{ik} - j^0 A^i = (\boldsymbol{E} \times \boldsymbol{B})_i - \rho\, \boldsymbol{A}\,. \end{aligned} \tag{5.29}$$

Likewise, Θ^{ij} would give the Maxwell stress tensor. The corresponding charges are the components of the four momentum $P^\mu = \int d^3x\, \Theta^{0i}$. Therefore, in the absence of an external current j^μ, the energy density of the electromagnetic field is $\frac{1}{2}\, (\boldsymbol{E}^2 + \boldsymbol{B}^2)$, its momentum density is the Poynting vector $\boldsymbol{E} \times \boldsymbol{B}$, and its spin density is $\boldsymbol{E} \times \boldsymbol{A}$.

Gauge fixing. Gauge invariance poses new problems for the quantization of the electromagnetic field. The field carries spin 1 and is of bosonic nature, so in principle we should impose the commutator relations

$$[A^\mu(x), \Pi^\nu(y)]_{x^0 = y^0} = i g^{\mu\nu}\, \delta^3(\boldsymbol{x} - \boldsymbol{y})\,. \tag{5.30}$$

Unfortunately this gives a contradiction because $\Pi^0 = 0$ vanishes and cannot have a non-trivial commutator with A^0. This reflects the redundancy that is inherent in the field A^μ. Gauge invariance tells us that we should restrict ourselves to a subset of fields A^μ that satisfy a certain gauge-fixing condition, for example

- the **Lorenz gauge** $\partial_\mu A^\mu = 0$: it only partially fixes the gauge, because we are still free to perform a *residual* gauge transformation $A'^\mu = A^\mu + \partial^\mu \varepsilon$ as long as $\Box\, \varepsilon = 0$. The Maxwell equations in the Lorenz gauge simply become $\Box\, A^\mu = j^\mu$.
- the **radiation gauge** $A^0 = 0$, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$: here the gauge fixing is complete because the remaining gauge parameter ε can be only a constant. The radiation gauge implies the Lorenz gauge $\partial_\mu A^\mu = 0$ but it is more restrictive.

There are two possible strategies for quantizing the theory: we could either fix the gauge in advance and thereby eliminate the unphysical degrees of freedom. The price we have to pay is the loss of manifest Lorentz covariance, and we have to check at the end of the quantization procedure that Lorentz symmetry is still intact. The second option is to work with the full gauge field A^μ and start from a *modified* Lagrangian where Π^0 does not vanish. This will introduce spurious degrees of freedom which we have to eliminate at the end.

We follow the second avenue and start from the following 'gauge-fixed' Lagrangian, where the gauge-fixing condition $\partial_\mu A^\mu = 0$ is implemented in the form of a Lagrange multiplier:

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\,(\partial\cdot A)^2\,. \tag{5.31}$$

The resulting equations of motion become

$$\partial_\mu\, F^{\mu\nu} + \lambda\,\partial^\nu\partial_\mu A^\mu = 0 \qquad \Leftrightarrow \qquad \Box\, A^\mu + (\lambda-1)\,\partial^\mu\partial_\nu A^\nu = 0\,. \tag{5.32}$$

Taking their divergence yields $\lambda\,\Box\,\partial_\mu A^\mu = 0$, which means that $\partial_\mu A^\mu$ must be a free scalar field that satisfies the massless Klein-Gordon equation. The additional term in the Lagrangian ensures that $\Pi^0 = -\lambda\,\partial_\mu A^\mu$ is no longer zero (the spatial components Π^i are unchanged), so in principle we can proceed with the quantization. Although we could discuss what follows for general λ, we set $\lambda = 1$ (**Feynman gauge**) because this simulates the Lorenz gauge condition in the Maxwell equations: $\Box\, A^\mu = 0$. (The limit where $\lambda\to\infty$ at the end of all calculations is called Landau gauge.) Following the steps in Eq. (5.10), it is easy to show that the action obtained from the Lagrangian (5.31) with $\lambda = 1$ is equivalent to that of the **Fermi Lagrangian**

$$\mathcal{L} = -\frac{1}{2}\,(\partial_\mu A_\nu)(\partial^\mu A^\nu)\,. \tag{5.33}$$

Its canonical conjugate momentum is

$$\Pi_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\nu)} = -\partial_0 A_\nu = -\dot{A}_\nu\,, \tag{5.34}$$

and the Hamiltonian of this theory becomes

$$H = \int d^3x\,\Big(\Pi_\nu \dot{A}^\nu - \mathcal{L}\Big) = \int d^3x\left[-\frac{1}{2}\,\dot{A}^2 - \frac{1}{2}\,(\boldsymbol{\nabla} A_\nu)(\boldsymbol{\nabla} A^\nu)\right]. \tag{5.35}$$

Polarization vectors. The solutions of the free Maxwell equations have the form

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\sum_{\lambda=0}^{3}\Big(a_{p,\lambda}\,\epsilon^\mu_{p,\lambda}\,e^{-ipx} + a^\dagger_{p,\lambda}\,\epsilon^{*\mu}_{p,\lambda}\,e^{ipx}\Big)\,, \tag{5.36}$$

which is compatible with $\Box\, A^\mu = 0$ as an operator equation as long as the four-vector p^μ is lightlike: $p^2 = 0 \Leftrightarrow E_p = |\boldsymbol{p}|$. The 4 linearly independent polarization vectors $\epsilon^\mu_{p,\lambda} = \epsilon^\mu(\boldsymbol{p},\lambda)$ depend on p^μ, and they can always be chosen to satisfy the following orthogonality and completeness relations:

$$\epsilon_{p,\lambda}\cdot\epsilon_{p,\lambda'} = g_{\lambda\lambda'}\,, \qquad \sum_{\lambda\lambda'} g^{\lambda\lambda'}\epsilon^\mu_{p,\lambda}\,\epsilon^\nu_{p,\lambda'} = g^{\mu\nu}\,. \tag{5.37}$$

The first relation implies that $\epsilon^2_{p,0} = 1$ and $\epsilon^2_{p,i} = -1$ so that $\epsilon^\mu_{p,0}$ is timelike whereas the others with $i = 1,2,3$ are spacelike. In particular, without going into a specific reference frame one can proceed as follows:

- The most general timelike vector that satisfies $n^2 = 1$ can be written in the form $n = (\sqrt{1+\boldsymbol{n}^2}, \boldsymbol{n})^T$. Therefore, set $\varepsilon^\mu_{p,0} = n^\mu$ for the **timelike** polarization. The remaining $\varepsilon^\mu_{p,i}$ must be transverse to n^μ with $n \cdot \varepsilon_{p,i} = 0$.
- Choose $\epsilon^\mu_{p,1}$ and $\epsilon^\mu_{p,2}$ **transverse** to p^μ, so that $p \cdot \varepsilon_{p,i} = 0$ for $i = 1, 2$.
- The remaining polarization vector $\epsilon^\mu_{p,3}$ must be a linear combination of n^μ and p^μ. The conditions $n \cdot \varepsilon_{p,3} = 0$ and $\epsilon^2_{p,3} = -1$ fix it uniquely: $\varepsilon^\mu_{p,3} = p^\mu/(p \cdot n) - n^\mu$. We call it the **longitudinal** polarization.

For example, with $\boldsymbol{p}$ in $z-$direction and $\boldsymbol{n} = 0$ this implies

$$p = |\boldsymbol{p}| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \varepsilon^\mu_{p,\,0\dots 3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.38}$$

Canonical quantization. Expressed in terms of the conjugate momentum (5.34), the commutation relations (5.30) take the form

$$\begin{aligned} [A^\mu(x), \dot{A}^\nu(x)]_{x^0=y^0} &= -ig^{\mu\nu}\, \delta^3(\boldsymbol{x} - \boldsymbol{y})\,, \\ [A^\mu(x), A^\nu(y)]_{x^0=y^0} &= 0\,, \\ [\dot{A}^\mu(x), \dot{A}^\nu(y)]_{x^0=y^0} &= 0\,. \end{aligned} \tag{5.39}$$

Note that the spatial components behave like ordinary scalar fields with respect to the commutator relation, whereas the sign for the timelike component is reversed. To extract the commutation relations for the ladder operators we can simply copy the steps from Eqs. (2.6–2.11) for the scalar field; the result is

$$[a_{p,\lambda}, a^\dagger_{p',\lambda'}] = -2E_p\, g_{\lambda\lambda'}\, \delta^3(\boldsymbol{p} - \boldsymbol{p}') \tag{5.40}$$

with all other commutators zero. Likewise, the momentum operator turns out to be

$$\begin{aligned} P^\mu &= -\int \frac{d^3p}{2E_p}\, p^\mu \sum_{\lambda\lambda'} g^{\lambda\lambda'}\, a^\dagger_{p,\lambda}\, a_{p,\lambda'} \Big|_{p^0=E_p=|\boldsymbol{p}|} \\ &= \int \frac{d^3p}{2E_p}\, p^\mu \left[-\, a^\dagger_{p,0}\, a_{p,0} + \sum_{\lambda=1}^{3} a^\dagger_{p,\lambda}\, a_{p,\lambda} \right]_{p^0=E_p=|\boldsymbol{p}|}. \end{aligned} \tag{5.41}$$

Also here the spatial modes have a positive sign but the timelike component comes with a minus. The number operator has an analogous form,

$$\widehat{N} = \int \frac{d^3p}{2E_p} \left[-\, a^\dagger_{p,0}\, a_{p,0} + \sum_{\lambda=1}^{3} a^\dagger_{p,\lambda}\, a_{p,\lambda} \right]_{p^0=E_p=|\boldsymbol{p}|}. \tag{5.42}$$

Despite appearances, the minus sign does not imply negative eigenvalues for these operators because when they act on a state $a^\dagger_{k,0}\,|0\rangle$ the sign cancels with that in the commutator relation:

$$P^\mu\, a^\dagger_{k,0}\,|0\rangle = k^\mu\, a^\dagger_{k,0}\,|0\rangle\,, \qquad \widehat{N}\, a^\dagger_{k,0}\,|0\rangle = a^\dagger_{k,0}\,|0\rangle\,. \tag{5.43}$$

But this is exactly the problem: the one-particle states $a^\dagger_{k,0}\,|0\rangle$ with timelike polarization $\lambda = 0$ have a *negative norm*,

$$\langle 0|\, a_{q,0}\, a^\dagger_{k,0}|0\rangle = -2E_k\,\delta^3(\boldsymbol{k}-\boldsymbol{q})\,, \tag{5.44}$$

which spoils the unitarity of the theory. How can we resolve this?

Gupta-Bleuler method. So far our quantization procedure is incomplete because we have not yet implemented the constraint $\partial_\mu A^\mu = 0$. It is impossible to impose it as an operator equation for the fields, because this would contradict our commutator relations:

$$0 \stackrel{!}{=} [A^\mu(x), \partial_\nu A^\nu(y)]_{x^0=y^0} = [A^\mu(x), \dot{A}^0(y)]_{x^0=y^0} = -ig^{\mu 0}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \neq 0\,. \tag{5.45}$$

What we can do instead is to implement it not at the level of the fields, but rather as a restriction on the Hilbert space. Let's decompose the field $A^\mu(x)$ into positive- and negative-frequency modes

$$\begin{aligned} A^\mu_+(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{2E_p}\sum_{\lambda=0}^{3} a_{p,\lambda}\,\epsilon^\mu_{p,\lambda}\,e^{-ipx}, \\ A^\mu_-(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{2E_p}\sum_{\lambda=0}^{3} a^\dagger_{p,\lambda}\,\epsilon^{*\mu}_{p,\lambda}\,e^{ipx}, \end{aligned} \tag{5.46}$$

so that $A^\mu = A^\mu_+ + A^\mu_-$. We say that the **physical states** $|\psi\rangle \in \mathbb{H}_{\text{phys}}$ are those states that satisfy the Gupta-Bleuler condition

$$\partial\cdot A_+\,|\psi\rangle \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad \langle\psi|\,\partial\cdot A_- = 0\,. \tag{5.47}$$

The two conditions are equivalent because $A_+^\dagger = A_-$, and taken together they imply that the classical constraint $\partial\cdot A = 0$ is now realized in the form of an expectation value:

$$\langle\psi|\,\partial\cdot A\,|\psi\rangle = \langle\psi|\,\partial\cdot A_+ + \partial\cdot A_-\,|\psi\rangle = 0\,. \tag{5.48}$$

We can work out the consequences of this relation by writing $\partial\cdot A_+$ in Fourier modes. According to our construction of the polarization vectors, their contraction with the lightlike momentum p^μ gives

$$p_\mu\,\epsilon^\mu_{p,\lambda} = \begin{cases} p\cdot n & \lambda = 0 \\ 0 & \lambda = 1,2 \\ -p\cdot n & \lambda = 3\,, \end{cases} \tag{5.49}$$

and therefore

$$\partial\cdot A_+ = -i\int\frac{d^3p}{2E_p}\,e^{-ipx}\sum_\lambda a_{p,\lambda}\,p_\mu\,\epsilon^\mu_{p,\lambda} = -i\int\frac{d^3p}{2E_p}\,e^{-ipx}\,p\cdot n\,(a_{p,0}-a_{p,3})\,. \tag{5.50}$$

Hence, the condition (5.47) for physical states $|\psi\rangle$ is equivalent to the condition

$$a_{p,0}\,|\psi\rangle \stackrel{!}{=} a_{p,3}\,|\psi\rangle\,. \tag{5.51}$$

Now observe that whenever we evaluate expectation values of operators of the form (5.41) or (5.42), we arrive at

$$\langle\psi|\, a^\dagger_{p,0}\, a_{p,0} - a^\dagger_{p,3}\, a_{p,3}\,|\psi\rangle = \langle\psi|\,(a^\dagger_{p,0} - a^\dagger_{p,3})\, a_{p,3}\,|\psi\rangle = 0\,. \tag{5.52}$$

Therefore, the timelike and longitudinal photons cancel each other in matrix elements, and only the transverse, physical polarizations $\lambda = 1, 2$ survive:

$$\langle\psi|\left[-a^\dagger_{p,0}\, a_{p,0} + \sum_{\lambda=1}^{3} a^\dagger_{p,\lambda}\, a_{p,\lambda}\right]|\psi\rangle = \langle\psi|\sum_{\lambda=1}^{2} a^\dagger_{p,\lambda}\, a_{p,\lambda}\,|\psi\rangle\,. \tag{5.53}$$

Physical state space. Let's find out what this means for a 'physical' one-particle state. We start by writing it as the most general superposition of polarization states with momentum $\boldsymbol{k}$:

$$|\psi\rangle = \sum_\lambda c_\lambda\, a^\dagger_{k,\lambda}\,|0\rangle\,. \tag{5.54}$$

Applying the condition (5.51) to it entails

$$(a_{p,0} - a_{p,3})\,|\psi\rangle = \sum_\lambda c_\lambda \underbrace{(a_{p,0} - a_{p,3})\, a^\dagger_{k,\lambda}\,|0\rangle}_{-2E_p\,\delta^3(\boldsymbol{p}-\boldsymbol{k})\,(g_{\lambda 0}-g_{\lambda 3})\,|0\rangle} \stackrel{!}{=} 0\,, \tag{5.55}$$

and therefore $c_0 = -c_3$, whereas c_1 and c_2 are unconstrained. This means there are two types of 'physical states' $|\psi\rangle$ that satisfy the transversality condition:

$$|\psi_T\rangle = \left(c_1\, a^\dagger_{k,1} + c_2\, a^\dagger_{k,2}\right)|0\rangle\,, \qquad |\phi\rangle = \left(a^\dagger_{k,0} - a^\dagger_{k,3}\right)|0\rangle\,, \tag{5.56}$$

whereas the negative-norm state $a^\dagger_{k,0}\,|0\rangle$ does *not* satisfy the constraint. On the other hand, a massless photon has only two physical polarizations, so what is the meaning of the state $|\phi\rangle$? Consider the scalar product

$$\langle\psi|\phi\rangle = \langle\psi|\left(a^\dagger_{k,0} - a^\dagger_{k,3}\right)|0\rangle = 0\,, \tag{5.57}$$

which must be zero because of Eq. (5.51). Since this holds for all states $|\psi\rangle$, and $|\phi\rangle$ is also one of them, it implies in particular $\langle\phi|\phi\rangle = 0$, i.e., the state $|\phi\rangle$ has zero norm. Because $|\phi\rangle$ is orthogonal to all $|\psi\rangle$, all scalar products of a general state $|\psi_T\rangle + c\,|\phi\rangle$ with any other physical state are the same as those with $|\psi_T\rangle$ alone, and therefore $|\phi\rangle$ decouples from all physical processes. In particular, it does not contribute to any matrix elements such as in Eq. (5.53), which are obtained from the transverse states $|\psi_T\rangle$ only: $\langle\phi|\,\mathcal{O}\,|\phi\rangle = 0$. States that decouple from the physics are also called **spurious**.

The decoupling statement will become nontrivial in the presence of interactions. As long as the interactions satisfy gauge invariance, the spurious states decouple from the $S-$matrix in all *external* legs where only the two transverse polarizations survive (this is a consequence of the Ward identities). However, the spurious states still contribute *internally* in the sense of virtual particles, where they are necessary to preserve unitarity.