7 Perturbation theory

The goal in the following is to calculate the n-point Green functions

$$\langle \Omega \,|\mathsf{T}\, \Phi(x_1) \dots \Phi(x_n) \,|\Omega\rangle \tag{7.1}$$

of a scalar theory in perturbation theory. That is, we assume that the interactions contained in $\mathcal{L}_{\text{int}} = \mathcal{L} - \mathcal{L}_0$ or $H_{\text{int}} = H - H_0$ are so weak that we can systematically expand these Green functions (and therefore also scattering amplitudes) in powers of the coupling constant. Perturbation theory has turned out to be immensely successful in many different contexts such as QED, the weak interaction, (to some extent) QCD, or also effective field theories.

Correlators in the interaction picture. How can we rearrange Eq. (7.1) in a form where $\mathcal{L}_{\text{int}}$ appears explicitly? To begin with, recall Eq. (2.61) which follows from translation invariance and tells us how the field $\Phi(x)$ evolves in time:

$$\Phi(t, \boldsymbol{x}) = e^{iH(t-t_0)}\, \Phi(t_0, \boldsymbol{x})\, e^{-iH(t-t_0)}\,. \tag{7.2}$$

$\Phi(x)$ is the field operator in the Heisenberg picture and carries the full spacetime dependence. Now let's define the **interaction picture field** $\Phi_I(x)$ as a field that 'evolves' with the Hamiltonian H_0 of the free theory:

$$\Phi_I(t, \boldsymbol{x}) = e^{iH_0(t-t_0)}\, \Phi_I(t_0, \boldsymbol{x})\, e^{-iH_0(t-t_0)}\,. \tag{7.3}$$

By definition this is a free field that satisfies the free KG equation with mass m_0, and we can expand it into Fourier modes as in Eq. (1.16). We now assume that the two fields are equal at some time t_0, where they have the same functional form $\Phi(t_0, \boldsymbol{x}) = \Phi_I(t_0, \boldsymbol{x})$. In that case we can relate $\Phi(x)$ and $\Phi_I(x)$ at arbitrary time $x_0 = t$ by

$$\Phi(t, \boldsymbol{x}) = U^\dagger(t, t_0)\, \Phi_I(t, \boldsymbol{x})\, U(t, t_0)\,, \qquad U(t, t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}\,. \tag{7.4}$$

Note that $U(t, t_0) \neq e^{-iH_{\text{int}}(t-t_0)}$ because H does not commute with H_0.

Actually the assumption $\Phi(t_0, \boldsymbol{x}) = \Phi_I(t_0, \boldsymbol{x})$ cannot hold in general, because Haag's theorem states that a free field will always remain free. That is, there is no unitary transformation that relates Φ to Φ_I, and consequently the interaction picture does not exist. We will ignore this problem in the following and hope that everything we do can still be justified in the sense of weakly converging matrix elements.

In any case, we can derive the following Schrödinger equation for the evolution operator $U(t, t_0)$:

$$\begin{aligned} i\frac{\partial U}{\partial t} &= e^{iH_0(t-t_0)}\, (H - H_0)\, e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}\, H_{\text{int}}\, e^{-iH_0(t-t_0)}\, U(t, t_0) =: H_I(t)\, U(t, t_0)\,. \end{aligned} \tag{7.5}$$

$H_I(t)$ is the Hamiltonian in the interaction picture, i.e., H_{int} evolved with H_0. It is simple because the functional dependence of $H_I(t)$ on $\Phi_I(t, \boldsymbol{x})$ is the same as that of H_{int} on $\Phi(t_0, \boldsymbol{x})$, for example in Φ^4 theory:

$$H_{\text{int}}(t_0) = \int d^3x\, \frac{\lambda}{4!}\, \Phi(t_0, \boldsymbol{x})^4 \quad \Rightarrow \quad H_I(t) = \int d^3x\, \frac{\lambda}{4!}\, \Phi_I(t, \boldsymbol{x})^4\,. \tag{7.6}$$

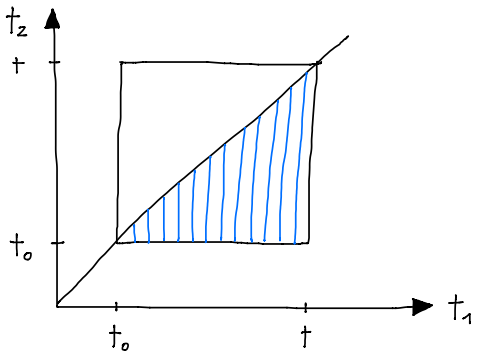


FIGURE 7.1: Symmetric integration domain in Eq. (7.7).

Therefore, the solution of (7.5) allows us to express the full field $\Phi(t, \boldsymbol{x})$ in terms of the interaction-picture Hamiltonian $H_I(t)$ and ultimately the interaction-picture field $\Phi_I(t, \boldsymbol{x})$, which is simple to handle because it is a free field that can be expanded into Fourier modes.

Another remark is in order: the explicit form for $U(t, t_0)$ in Eq. (7.4) only holds for the case where H, H_0 and H_{int} are all time-independent. This is true for the full Hamiltonian H but in general not for $H_0(t)$ and $H_{\text{int}}(t)$: $\dot{H}_{\text{int}}(t) = i[H, H_{\text{int}}(t)] = i[H_0(t), H_{\text{int}}(t)]$. Fortunately, it is not necessary to specify $U(t, t_0)$ explicitly: one can show that the generic relation between $\Phi(x)$ and $\Phi_I(x)$ in Eq. (7.4) leads to the same Schrödinger equation.

Eq. (7.5) is solved by

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 \, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) \, H_I(t_2) + \dots \tag{7.7}$$

To see this, take the time derivative with $(\partial/\partial t) \int_{t_0}^{t} dt' f(t') = f(t)$: each term in the series reproduces the previous one with a factor $-iH_I(t)$, and the initial condition $U(t_0, t_0) = 1$ is satisfied. Note that the factors H_I in the integrand are automatically time-ordered because $t_1 > t_2 > t_3 > \dots$, so we can equally write

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 \, H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, \mathsf{T} \{H_I(t_1) \, H_I(t_2)\} + \dots \tag{7.8}$$

Here we additionally exploited the fact that the integral is symmetric in t_1 and t_2, cf. Fig. 7.1; this also holds for the higher-order diagrams. The series defines the **time-ordered exponential**

$$U(t, t_0) =: \mathsf{T} \exp \left[-i \int_{t_0}^{t} dt' \, H_I(t') \right] \tag{7.9}$$

as the time ordering of the individual terms in the series expansion. From the expansion one can also prove the properties

$$\begin{aligned} &U^\dagger(t_1,t_2) = U^{-1}(t_1,t_2) = U(t_2,t_1)\,, \\ &U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3) \quad \text{for} \quad t_1 \geq t_2 \geq t_3\,. \end{aligned} \tag{7.10}$$

The hermitian conjugation switches all i factors and can be reversed by exchanging the integration limits, which leads to the first relation. To verify the second one, observe that $U(t,t_2)\,U(t_2,t_3)$ satisfies the same Schrödinger equation (7.5) so it can be written as $U(t,t')$, and the boundary condition $U(t_3,t') = 1$ entails $t' = t_3$.

With Eqs. (7.4) and (7.10) at hand, we can work out the full two-point function. Consider first the case $x^0 > y^0$:

$$\begin{aligned} \langle\Omega|\,\Phi(x)\,\Phi(y)\,|\Omega\rangle &= \langle\Omega|\,U^\dagger(x_0,t_0)\,\Phi_I(x)\,U(x_0,t_0)\,U^\dagger(y_0,t_0)\,\Phi_I(y)\,U(y_0,t_0)\,|\Omega\rangle \\ &= \langle\Omega|\,U(t_0,x_0)\,\Phi_I(x)\,U(x_0,y_0)\,\Phi_I(y)\,U(y_0,t_0)\,|\Omega\rangle\,. \end{aligned} \tag{7.11}$$

Let's insert some large time $T \gg x^0, y^0$:

$$\cdots = \langle\Omega|\,U(t_0,T)\,\underbrace{U(T,x_0)\,\Phi_I(x)\,U(x_0,y_0)\,\Phi_I(y)\,U(y_0,-T)}_{\text{time-ordered}}\,U(-T,t_0)\,|\Omega\rangle\,. \tag{7.12}$$

The bracket is then already time-ordered, so we can put a time-ordering symbol in front of it and combine all U's inside:

$$\cdots = \langle\Omega|\,U(t_0,T)\,\mathsf{T}\Big\{\Phi_I(x)\,\Phi_I(y)\,U(T,-T)\Big\}\,U(-T,t_0)\,|\Omega\rangle\,. \tag{7.13}$$

Since this is time-ordered, the opposite case with $y^0 > x^0$ gives the same result, and therefore the full correlator becomes

$$\langle\Omega|\,\mathsf{T}\,\Phi(x)\,\Phi(y)\,|\Omega\rangle = \langle\Omega|\,U(t_0,T)\,\mathsf{T}\Big\{\Phi_I(x)\,\Phi_I(y)\,U(T,-T)\Big\}\,U(-T,t_0)\,|\Omega\rangle\,. \tag{7.14}$$

The quantity $U(T,-T)$ is given by

$$U(T,-T) = \mathsf{T}\,\exp\left[-i\int_{-T}^{T} dt\,H_I(t)\right] = \mathsf{T}\,e^{iS_I}\,, \tag{7.15}$$

where $S_I = \int d^4x\,\mathcal{L}_I = -\int d^4x\,\mathcal{H}_I$ is the action corresponding to the interacting part that depends on the field $\Phi_I(x)$. We assumed that the interacting Lagrangian contains no field derivatives so that $\mathcal{L}_I = -\mathcal{H}_I$. The expression (7.14) still depends on the arbitrary reference time t_0 and the interacting vacuum $|\Omega\rangle$ which we have to get rid of.

Free vs. interacting vacuum. We would like to relate the full interacting vacuum $|\Omega\rangle$ to the vacuum $|0\rangle$ of the free theory. To do so, recall that $H\,|\Omega\rangle = 0$ and $\langle\Omega|\Omega\rangle = 1$. That is, we 'renormalized' the interacting theory so that the vacuum energy $E_\Omega = 0$, which we motivated with the arbitrary counterterm V_0 in the Lagrangian. However, doing so removes our freedom to set the vacuum energy in the corresponding free theory

(defined by H_0) to zero: $H_0 |n\rangle = E_n |n\rangle$ with $E_0 \leq E_1 \leq E_2 \leq \ldots$, but $E_0 \neq 0$. In any case we can write

$$U(-T, t_0) |\Omega\rangle = e^{-iH_0(T+t_0)} \underbrace{e^{iH(T+t_0)} |\Omega\rangle}_{=|\Omega\rangle} = \sum_{n=0}^{\infty} e^{-iE_n(T+t_0)} |n\rangle\langle n|\Omega\rangle \,.$$

In the last step we inserted a complete set of states of the free theory. Assuming that $\langle 0|\Omega\rangle \neq 0$, we can eliminate the contributions from the states with higher energy by taking the limit $T \to \infty(1 - i\epsilon)$ because this will eliminate all contributions from the energies $E_n > E_0$:

$$U(-T, t_0) |\Omega\rangle \xrightarrow{T\to\infty(1-i\epsilon)} e^{-iE_0(T+t_0)} |0\rangle\langle 0|\Omega\rangle =: c(-T, t_0) |0\rangle \,.$$

The analogous case for $\langle\Omega|$ gives

$$\langle\Omega| U^\dagger(T, t_0) \xrightarrow{T\to\infty(1-i\epsilon)} \langle 0| c^*(T, t_0) \,. \tag{7.16}$$

After plugging this into Eq. (7.14) we are still left with the awkward factors $c(-T, t_0)$ and $c^*(T, t_0)$. We can remove them too by noting that

$$\begin{aligned} \langle\Omega|\Omega\rangle = 1 \xrightarrow{T\to\infty(1-i\epsilon)} & c^*(T, t_0)\, c(-T, t_0)\, \langle 0| U(T, t_0)\, U^\dagger(-T, t_0) |0\rangle \\ & = c^*(T, t_0)\, c(-T, t_0)\, \langle 0| U(T, -T) |0\rangle \,. \end{aligned} \tag{7.17}$$

Inserting everything into Eq. (7.14) we arrive at the final result for the full propagator:

$$\langle\Omega| \mathsf{T}\, \Phi(x)\, \Phi(y) |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| \mathsf{T} \left\{ \Phi_I(x)\, \Phi_I(y)\, U(T, -T) \right\} |0\rangle}{\langle 0| U(T, -T) |0\rangle} \,. \tag{7.18}$$

It can be generalized to arbitrary $n-$point functions:

$$\langle\Omega| \mathsf{T}\, \Phi(x_1) \ldots \Phi(x_n) |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| \mathsf{T} \left\{ \Phi_I(x_1) \ldots \Phi_I(x_n)\, e^{iS_I} \right\} |0\rangle}{\langle 0| \mathsf{T}\, e^{iS_I} |0\rangle} \,. \tag{7.19}$$

With this formula we have in principle everything in place to do perturbation theory. We could expand e^{iS_I} in the small coupling constant, express Φ_I in terms of creation and annihilation operators (since it is a free field), take the time ordering, and calculate any correlation function simply by brute force. However, this also becomes quite repetitive and cumbersome, which is where Wick's theorem comes to rescue.

Wick's theorem. To shorten the notation, we will write the interaction-picture field as $\Phi_I(x) = \phi(x)$. Since it is a free field, we can decompose it into positive- and negative-frequency parts:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a_p\, e^{-ipx} + a_p^\dagger\, e^{ipx} \right) = \phi_+(x) + \phi_-(x) \,, \tag{7.20}$$

with $\phi_+(x) |0\rangle = 0 = \langle 0| \phi_-(x)$. In the following we want to express products of field operators in terms of their normal ordered versions, which means that all creation operators are shuffled to the left and all annihilation operators to the right or, equivalently, all instances of $\phi_-(x)$ go to the left and all instances of $\phi_+(x)$ to the right.

Consider the product of two fields $\phi(x)\,\phi(y)$. In terms of positive- and negative-frequency modes it has the form

$$\begin{aligned}\phi(x)\,\phi(y) &= \phi_+(x)\,\phi_+(y) + \phi_+(x)\,\phi_-(y) + \phi_-(x)\,\phi_+(y) + \phi_-(x)\,\phi_-(y) \\ &= :\phi(x)\,\phi(y): + [\phi_+(x), \phi_-(y)]\,,\end{aligned} \tag{7.21}$$

and likewise $\phi(y)\,\phi(x) = :\phi(x)\,\phi(y): + [\phi_+(y), \phi_-(x)]$. Inserting the Fourier modes, we find for the commutator

$$[\phi_+(x), \phi_-(y)] = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_p}\, e^{-ip(x-y)}\Big|_{p^0=E_p} = D(x-y)\,, \tag{7.22}$$

where $D(x-y)$ has been given in Eq. (2.72). In total this yields

$$\begin{aligned}\mathsf{T}\,\phi(x)\,\phi(y) &= :\phi(x)\,\phi(y): + \Theta(x^0-y^0)\,D(x-y) + \Theta(y^0-x^0)\,D(y-x) \\ &= :\phi(x)\,\phi(y): + D_F(x-y)\,.\end{aligned} \tag{7.23}$$

Since $\langle 0|:\mathcal{O}:|0\rangle = 0$, this implies for the vacuum expectation value

$$\langle 0|\,\mathsf{T}\,\phi(x)\,\phi(y)\,|0\rangle = D_F(x-y)\,, \tag{7.24}$$

which is just our earlier definition of the Feynman propagator (remember that $\phi(x)$ is a free field and $|0\rangle$ the free vacuum).

What is useful about the identity is that we can immediately generalize it to arbitrary $n-$point functions. This is known as **Wick's theorem**, and it states that the time-ordered product $\mathsf{T}\,\phi(x_1)\ldots\phi(x_n)$ is equal to the normal-ordered product $:\phi(x_1)\ldots\phi(x_n):$ plus all possible combinations of normal orderings and *contractions* of distinct fields. A contraction of two fields $\phi(x_1)$, $\phi(x_2)$ is defined to be equal to the Feynman propagator $D(x_1-x_2)$ and denoted by

$$\overbrace{\phi(x)\,\phi(y)} = D_F(x-y)\,. \tag{7.25}$$

Using the shorthand notation $\phi(x_i) = \phi_i$ and $D_F(x_i - x_j) = D_{ij}$, let's illustrate the result for the four-point function:

$$\begin{aligned}\mathsf{T}\,\{\phi_1\,\phi_2\,\phi_3\,\phi_4\} = {}& :\phi_1\,\phi_2\,\phi_3\,\phi_4: \\ &+ D_{12}\,:\phi_3\,\phi_4: \;+ D_{13}\,:\phi_2\,\phi_4: \;+ D_{14}\,:\phi_2\,\phi_3: \\ &+ D_{23}\,:\phi_1\,\phi_4: \;+ D_{24}\,:\phi_1\,\phi_3: \;+ D_{34}\,:\phi_1\,\phi_2: \\ &+ D_{12}\,D_{34} + D_{13}\,D_{24} + D_{14}\,D_{23}\,.\end{aligned} \tag{7.26}$$

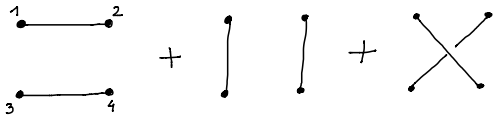
The Wick theorem for arbitrary $n-$point functions can be proven via induction (see Peskin-Schroeder, p.90). Only the last line above survives when taking vacuum expectation values, and therefore the VEV of a time-ordered product of fields equals the sum over all possible contractions:

$$\langle 0|\,\mathsf{T}\,\{\phi_1\,\phi_2\,\phi_3\,\phi_4\}\,|0\rangle = D_{12}\,D_{34} + D_{13}\,D_{24} + D_{14}\,D_{23}\,. \tag{7.27}$$

If n is odd, the VEV vanishes because there is always an odd number of normal-ordered fields remaining.

Feynman diagrams. A diagrammatic way to visualize such contractions is to draw Feynman diagrams: draw a point for each spacetime argument x_i and connect them by lines, which represent the Feynman propagators of the free theory. The four-point function from Eq. (7.27) then becomes

$$\langle 0|\,\mathsf{T}\,\{\phi_1\,\phi_2\,\phi_3\,\phi_4\}\,|0\rangle \;=$$



. (7.28)

More interesting are expressions that contain more than one field at the same spacetime point, which leads to **loop diagrams**. Let's put Eq. (7.18) for the two-point function in ϕ^4 theory to use. When we expand the exponential in the numerator to $\mathcal{O}(\lambda)$ we obtain

$$\begin{aligned}\langle 0|\,\mathsf{T}\,\phi(x)\,\phi(y)\,e^{-i\frac{\lambda}{4!}\int d^4z\,\phi(z)^4}|0\rangle &= \\ = \langle 0|\,\mathsf{T}\,\phi(x)\,\phi(y)\,|0\rangle - i\frac{\lambda}{4!}\,\langle 0|\,\mathsf{T}\,\phi(x)\,\phi(y)\int d^4z\,\phi(z)^4\,|0\rangle + \ldots &\end{aligned} \tag{7.29}$$

The first term is just the propagator line from x to y. Applying the Wick theorem to the combination $\phi(x)\phi(y)\phi(z)^4$ yields only two distinct expressions:

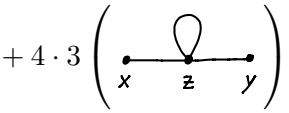
- If we contract $\phi(x)$ with $\phi(y)$, there are three distinguishable ways how to contract $\phi(z)$ with $\phi(z)$:
$$\overbracket{\phi(x)\,\phi(y)}\,\overbracket{\phi(z)\,\phi(z)}\,\overbracket{\phi(z)\,\phi(z)}\,.$$
- if we contract $\phi(x)$ with $\phi(z)$ (four possibilities) and $\phi(y)$ with $\phi(z)$ (three possibilities), there is one possibility left how to contract $\phi(z)$ with $\phi(z)$:
$$\overbracket{\phi(x)\,\overbracket{\phi(y)\,\phi(z)}\,\phi(z)}\,\overbracket{\phi(z)\,\phi(z)}\,.$$

In total, this gives

$$\langle 0|\,\mathsf{T}\big\{\phi_x\,\phi_y\,\phi_z\,\phi_z\,\phi_z\,\phi_z\big\}|0\rangle = 3\cdot D_{xy}\,D_{zz}\,D_{zz} + 4\cdot 3\cdot D_{xz}\,D_{yz}\,D_{zz}\,, \tag{7.30}$$

or in terms of diagrams:

$$\langle 0|\,\mathsf{T}\Big\{\phi_x\,\phi_y\int_z\phi_z\,\phi_z\,\phi_z\,\phi_z\Big\}|0\rangle \;=\; 3\,\Bigg($$

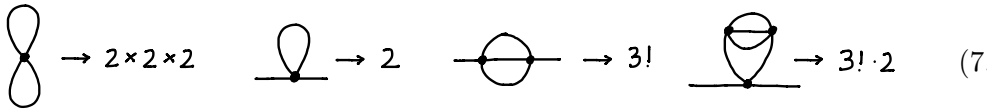


$$\Bigg) + 4\cdot 3\,\Bigg($$

$$\Bigg)\,, \tag{7.31}$$

where we abbreviated $\int d^4z = \int_z$.

Clearly, for higher products of fields the number of possible Wick contractions will rise dramatically. Fortunately, however, this number *almost* cancels with the factors 4! from the denominators in the Taylor expansion. In the example above, the final prefactors are 1/8 and 1/2. Their denominators 8 and 2 are called **symmetry factors** of the diagrams, because they count the number of possibilities to exchange the components without changing the diagram itself. For example:



(7.32)

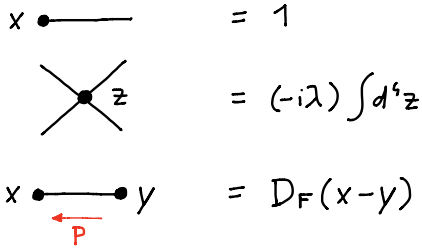
In the first diagram we can flip both the upper and the lower loop horizontally, and we can exchange the loops vertically, which gives $2 \times 2 \times 2 = 8$. In the second diagram we can only do a horizontal flip, so the symmetry factor is 2. In the third diagram there are $3! = 6$ possibilities to exchange the three internal lines, and in the fourth diagram we can additionally perform a horizontal flip ($3! \times 2 = 12$). Note that the external points x and y are fixed and cannot be flipped.

Feynman rules. These observations hold in general and can be summarized by the Feynman rules. Consider an $n-$point function for a theory with a ϕ^m interaction:

$$\langle 0|\, \mathsf{T}\, \phi(x) \dots \phi(x_n)\, e^{-i\frac{\lambda}{m!}\int d^4z\, \phi(z)^m} |0\rangle\,. \tag{7.33}$$

You can find all diagrams at a given order $\mathcal{O}(\lambda^k)$ in perturbation theory if you draw

- n external points x_i,
- k internal points z_j (**vertices**) with m incoming lines,
- connect all points by Feynman propagators of the free theory,
- divide each diagram by its symmetry factor,
- and sum up all diagrams in the end.

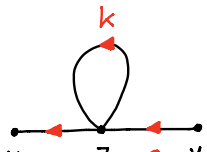


It is usually more convenient to write the Feynman propagator in momentum space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, D_F(p)\,, \qquad D_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon}\,. \tag{7.34}$$

We will use the convention that the momentum points from right to left, i.e., from y to x. This is irrelevant for a scalar propagator because $D_F(x-y) = D_F(y-x)$, but the distinction will become important when we generalize the Feynman rules to fermions.

As an example, let's work out the 'tadpole' diagram, now with the abbreviation $\int_p = \int \frac{d^4p}{(2\pi)^4}$:

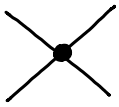


$$\begin{aligned}
&= \frac{1}{2}\,(-i\lambda) \int d^4z\, D_F(x-z)\, D_F(y-z)\, D_F(z-z) \\
&= -\frac{i\lambda}{2} \int d^4z \iiint_{p\,q\,k} e^{-ip(x-z)}\, e^{iq(y-z)}\, D_F(p)\, D_F(q)\, D_F(k) \\
&= -\frac{i\lambda}{2} \iiint_{p\,q\,k} e^{-ipx}\, e^{iqy}\, D_F(p)\, D_F(q)\, D_F(k)\, (2\pi)^4\, \delta^4(p-q)\,.
\end{aligned} \tag{7.35}$$

From here one can read off the **Feynman rules 'in momentum space'** (this is a bit of a misnomer because the Green function is still given in real space), which are easier to handle in practice:

•) 

$= e^{-ipx}$

•) 
$= -i\lambda\,(2\pi)^4\,\delta^4\Big(\sum p_i - \sum q_j\Big)$

•) 

$= D_F(p) = \dfrac{i}{p^2 - m_0^2 + i\varepsilon}$

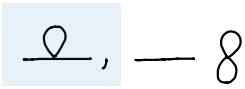
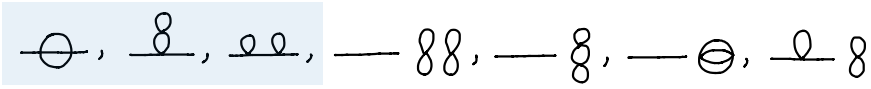
•) integrate over all momenta: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$

•) divide by the symmetry factor.

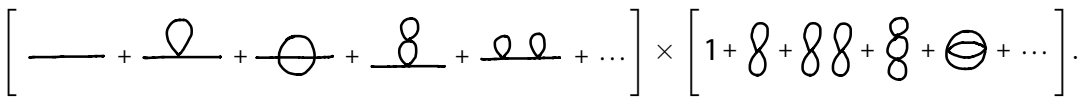
Propagator in ϕ^4 theory. Let's put the Feynman rules to use and calculate the propagator of ϕ^4 theory, i.e., all diagrams that contribute to

$$\langle 0|\,\mathsf{T}\,\phi(x)\,\phi(y)\,e^{-i\frac{\lambda}{m!}\int d^4z\,\phi(z)^m}|0\rangle \tag{7.36}$$

up to $\mathcal{O}(\lambda^k)$. The propagator has two external points x and y, and a diagram at $\mathcal{O}(\lambda^k)$ has k vertices. Here is the complete list up to $\mathcal{O}(\lambda^2)$:

- $\mathcal{O}(\lambda^0)$:
- $\mathcal{O}(\lambda^1)$: 
- $\mathcal{O}(\lambda^2)$: 

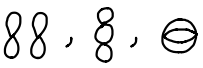
Observe that we arrive at the same result if we multiply the sum of all **connected diagrams** (those in the boxes) by the sum of all **vacuum bubbles:**



This is not a coincidence because so far we have only dealt with the numerator in Eq. (7.19). To obtain the full Green function, we should also take into account the denominator

$$\langle 0|\,\mathsf{T}\,e^{-i\frac{\lambda}{m!}\int d^4z\,\phi(z)^m}|0\rangle\,, \tag{7.37}$$

which is called the **partition function**. Its perturbative expansion generates just these vacuum bubbles:

$\mathcal{O}(\lambda^0)$: 1 $\mathcal{O}(\lambda^1)$:  $\mathcal{O}(\lambda^2)$: 

so they factor out in the full Green function. Therefore we find

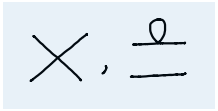
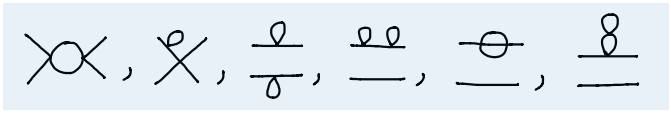
$$\langle\Omega|\,\mathsf{T}\,\Phi(x)\,\Phi(y)\,|\Omega\rangle = \sum \text{(partially) connected terms}\,. \tag{7.38}$$

The meaning of 'partially connected' will become clear in a moment.

Four-point function in ϕ^4 theory. As another example, let's have a look at the four-point function

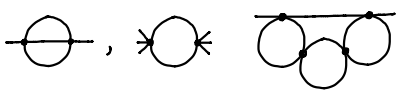
$$\frac{\langle 0|\,\mathsf{T}\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\,e^{-i\frac{\lambda}{m!}\int d^4z\,\phi(z)^m}|0\rangle}{\langle 0|\,\mathsf{T}\,e^{-i\frac{\lambda}{m!}\int d^4z\,\phi(z)^m}|0\rangle}\,. \tag{7.39}$$

It has four external points and k vertices at $\mathcal{O}(\lambda^k)$. Ignoring pure vacuum bubbles, the diagrams up to $\mathcal{O}(\lambda^2)$ are given by

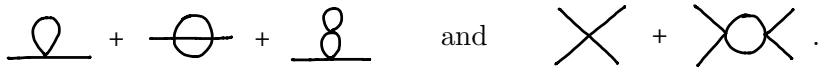
- $\mathcal{O}(\lambda^0)$: 
- $\mathcal{O}(\lambda^1)$: 
- $\mathcal{O}(\lambda^2)$: 

We already found the zeroth-order result in Eq. (7.28) as the sum of the three disconnected terms. Since they all have the same structure we have represented them here by a single diagram for brevity. The same goes for the other diagrams where we have only drawn one representative for each case, e.g. for the fourth diagram at $\mathcal{O}(\lambda^2)$: we can attach the two bubbles at the upper and lower line, and there are three permutations of the two lines. Here it also becomes clear why we referred to 'partially connected' terms in Eq. (7.38): the full Green function is the sum of those diagrams where we can no longer factor out vacuum bubbles, but they do not need to be fully connected.

1-particle irreducible diagrams. A class of diagrams that are important for theoretical analyses are the 1PI (one-particle irreducible) diagrams. The 1PI property is defined as follows: consider only diagrams which are fully connected. Remove ('amputate') its external legs. If the diagram is still connected after cutting a single internal line, it is 1PI. Some examples and counterexamples are:

1PI:  not 1PI: 

Up to $\mathcal{O}(\lambda^2)$, the 1PI contributions to the propagator and the four-point function are therefore the following:

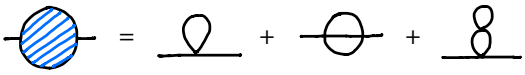


Now let's denote the full propagator by

$$G(x-y) = \langle\Omega|\,\mathsf{T}\,\Phi(x)\,\Phi(y)\,|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, G(p) \;=\; \tag{7.40}$$

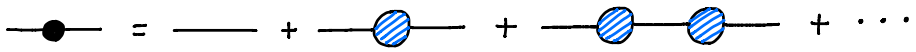
and define the **self-energy** of the scalar particle as the sum of all 1PI graphs for the 2-point function:

$$\frac{\Sigma(p)}{i} :=$$



Observe that we can obtain the propagator by resumming its 1PI contributions:

$$G(p) \;=$$



$$= D_F\left[1 + \frac{\Sigma}{i}\left(D_F + D_F\frac{\Sigma}{i}\,D_F + \dots\right)\right] \tag{7.41}$$

$$= D_F\left[1 + \frac{\Sigma}{i}\,G(p)\right], \tag{7.42}$$

and therefore

$$iG^{-1}(p) = p^2 - m_0^2 - \Sigma(p) \quad\Leftrightarrow\quad G(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p) + i\epsilon}\,. \tag{7.43}$$

On the other hand, we know from the Källén-Lehmann spectral representation (6.12) that the full propagator must have the form

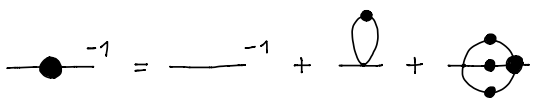
$$G(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} \;+\; \text{terms that are regular at } p^2 = m^2\,. \tag{7.44}$$

In this sense the self-interactions of the particle (the quantum loop corrections) shift its mass from m_0 to m, so that the pole appears at $p^2 = m^2$, and $\Sigma(p)$ takes indeed the meaning of a self-energy.

What we have done here is resumming the geometric series. For illustration, replace $\Sigma \to x$, $D_F \to i$ and $G \to if(x)$:

$$f(x) = 1 + x + x^2 + \dots = 1 + x\,(1 + x + \dots) = 1 + xf(x) \quad\Rightarrow\quad f(x) = \frac{1}{1-x}\,. \tag{7.45}$$

Of course this is only justified for $|x| < 1$, i.e., as long as the coupling is small. Fortunately, Eq. (7.43) can be also derived nonperturbatively: it is the Dyson-Schwinger equation for the propagator, which is an exact equation:



(7.46)

All ingredients with filled blobs are dressed. In our example this means that

$$f(x) = \frac{1}{1-x} = 1 + xf(x) = 1 + x + x^2 f(x) = \dots \tag{7.47}$$

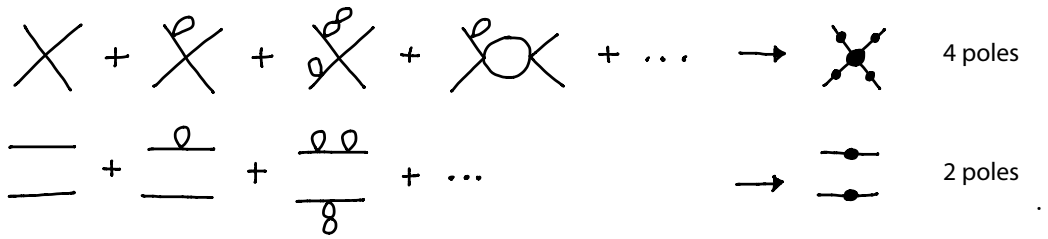
is valid for all x except $x = 1$, because there is always a remainder that reproduces the exact result, whereas the geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ converges to the exact value only for $|x| < 1$. Hence, the Dyson-Schwinger equation is more general than the perturbative expansion.

There may be also genuinely nonperturbative effects that are not reproducible by the perturbative series, not even for a small coupling. An example is QCD in the chiral limit, where the dressed propagator has a nonvanishing mass function even if the mass in the Lagrangian is zero. This effect is due to spontaneous chiral symmetry breaking; although it follows from the Dyson-Schwinger equation, it cannot be achieved at any order in perturbation theory.

In a similar way one can generate higher $n-$point functions from their 1PI counterparts because they can only differ by internal (fully resummed) propagator lines. Hence, the 1PI correlation functions encode the 'irreducible' content of an $n-$point interaction. In Sec. 8 we will also see that they are convenient for discussing the renormalization of the quantum field theory.

Scattering amplitude. With all that in mind, we can now go back to the scattering amplitude and the LSZ formula (6.35). There we found that the full Green function is proportional to the connected S-matrix element, with one pole of mass m attached for each external particle, plus further disconnected diagrams. We argued that the Green function will be a sum of terms with different pole factors, and only those terms survive in the S-matrix element where the number of poles matches exactly.

From the discussion above it is clear that such pole factors can only come from fully resummed propagators with mass m. This means that only **connected** terms in the Green function can contribute to the S-matrix, for example:



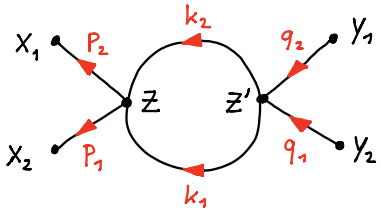
Since the external particles are onshell, removing the pole factors is equivalent to removing the dressed propagators according to Eq. (7.44). We ignore the remaining Z factors because in the process of renormalization we will absorb them into the fields. In that way we arrive at the final result for the S-matrix element expressed through the renormalized field $\Phi(x)$, which we write in terms of the **invariant amplitude** $\mathcal{M}$:

$$\begin{aligned}(2\pi)^4\,\delta^4\Big(\sum p_i - \sum q_j\Big)\,i\mathcal{M} :&= (2\pi)^{\frac{3}{2}(n+r)}\langle p_1 \dots p_n, \text{out}\,|\,q_1 \dots q_r, \text{in}\rangle_{\text{conn.}} \\ &= \text{FT}\ \langle\Omega|\,\mathsf{T}\,\Phi(x_1)\dots\Phi(y_r)\,|\Omega\rangle_{\substack{\text{connected,}\\ \text{amputated,}\\ \text{onshell}}} .\end{aligned} \tag{7.48}$$

Since all external momenta are onshell, they describe physical particles with $p_i^2 = m^2$. The internal propagators, whose loop momenta are integrated over, are offshell and correspond to **virtual particles** with $k^2 \neq m^2$ (note that four-momentum conservation

is still satisfied at each vertex). In this sense the scattering amplitude is the summation over all possible virtual processes that can contribute.

What remains to be done is to state the Feynman rules for the scattering matrix element. Let's derive them explicitly for the 1-loop graph at $\mathcal{O}(\lambda^2)$:



$$= I(x_1, x_2, y_1, y_2) \tag{7.49}$$

Employing the Feynman rules in momentum space, the diagram takes the form

$$\begin{aligned} \cdots = \frac{(-i\lambda)^2}{2} \int_{p_1}\int_{p_2}\int_{q_1}\int_{q_2}\int_{k_1}\int_{k_2} & (2\pi)^4\,\delta^4(p_1+p_2-k_1-k_2)\,(2\pi)^4\,\delta^4(k_1+k_2-q_1-q_2) \\ & \times e^{-ip_1x_1}\,e^{-ip_2x_2}\,e^{iq_1y_1}\,e^{iq_2y_2} \\ & \times D_F(p_1)\,D_F(p_2)\,D_F(q_1)\,D_F(q_2)\,D_F(k_1)\,D_F(k_2)\,. \end{aligned} \tag{7.50}$$

For the S-matrix element we need the amplitude in momentum space, so we take the Fourier transform

$$\begin{aligned} & \int d^4x_1\,e^{ip_1x_1}\int d^4x_2\,e^{ip_2x_2}\int d^4y_1\,e^{-iq_1y_1}\int d^4y_2\,e^{-iq_2y_2}\,I(x_1,x_2,y_1,y_2) \\ & = \frac{(-i\lambda)^2}{2}\,D_F(p_1)\,D_F(p_2)\,D_F(q_1)\,D_F(q_2) \\ & \times \underbrace{\int_{k_1}\int_{k_2}(2\pi)^4\,\delta^4(p_1+p_2-k_1-k_2)\,(2\pi)^4\delta^4(k_1+k_2-q_1-q_2)\,D_F(k_1)\,D_F(k_2)}_{=(2\pi)^4\,\delta^4(p_1+p_2-q_1-q_2)\int_k D_F(k)\,D_F(p_1+p_2-k)} \end{aligned} \tag{7.51}$$

By amputating the external propagators we obtain the contribution to the S-matrix element:

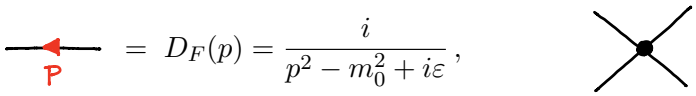
$$\begin{aligned} & \mathrm{FT}\,\{I(x_1,x_2,y_1,y_2)\}_{\text{amputated}} = \\ & \qquad = (2\pi)^4\,\delta^4(p_1+p_2-q_1-q_2)\,\frac{(-i\lambda)^2}{2}\int_k D_F(k)\,D_F(p_1+p_2-k) \\ & \qquad = (2\pi)^4\,\delta^4(p_1+p_2-q_1-q_2)\,i\mathcal{M}\,. \end{aligned} \tag{7.52}$$

The $\delta-$function reflects total momentum conservation; we already anticipated it when we defined the invariant amplitude via Eq. (7.48). Therefore, the result is simply

$$i\mathcal{M} = \frac{(-i\lambda)^2}{2}\int_k D_F(k)\,D_F(p_1+p_2-k)\,. \tag{7.53}$$

From this expression we can read off the **Feynman rules for S-matrix elements**, which become extremely simple because all external propagators have disappeared. For an $n-$point function in a Φ^m theory at $\mathcal{O}(\lambda^k)$,

- draw n external points and k vertices with m ingoing lines, and connect all lines;
- write the propagators and vertices as



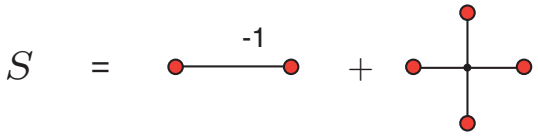
$$= D_F(p) = \frac{i}{p^2 - m_0^2 + i\varepsilon}\,, \qquad = -i\lambda\,, \tag{7.54}$$

  and impose momentum conservation at each vertex;
- integrate over all *loop* momenta $\int \frac{d^4k}{(2\pi)^4}$;
- divide by the symmetry factor of the diagram;
- set all external momenta onshell.

The two diagrams in Eq. (7.54) are the only elementary building blocks that we have at our disposal in a Φ^4 theory. In principle we can read them off directly from the Lagrangian:

$$\begin{aligned} S &= \int d^4x \left[\frac{1}{2}\,\partial_\mu\Phi\,\partial^\mu\Phi - \frac{1}{2}\,m_0^2\,\Phi^2 - \frac{\lambda}{m!}\,\Phi^m\right] \\ &\overset{p.I.}{\simeq} \int d^4x \left[-\frac{1}{2}\,\Phi\,(\Box + m_0^2)\,\Phi - \frac{\lambda}{m!}\,\Phi^m\right]. \end{aligned} \tag{7.55}$$

After taking a Fourier transform of each field, the Klein-Gordon operator becomes the inverse tree-level propagator $D_F^{-1}(p) = p^2 - m_0^2 + i\varepsilon$ in momentum space, and the tree-level interaction vertex follows from removing the fields together with the combinatorial factor 4!. In an extremely symbolic sense we could write the action (here for a Φ^4 theory) as



where the circles represent the fields $\Phi(x)$. Such a symbolic notation is indeed useful in the path-integral approach, where Green functions are obtained as functional derivatives of the classical action or the quantum effective action.

To summarize, the basic goal of a quantum field theory is to calculate the fully dressed $n-$point Green functions, including all quantum corrections, by starting from the tree-level expressions that are specified by the classical Lagrangian. These $n-$point functions are the quantities that enter scattering matrix elements from where we can extract observables.