Chapter 3

Hadrons

In the previous chapter we mostly ignored the flavor structure of the QCD Lagrangian because it was less relevant for the properties of quarks and gluons compared to the color structure. Vice versa, QCD's local gauge invariance does not tell us much about the systematics of the hadron spectrum except that hadrons are color singlets and can be constructed from $q\bar{q}$ and qqq (and also more complicated combinations). Here we will turn the wheel around and focus exclusively on the global flavor symmetries of QCD, in particular chiral symmetry, which becomes important and leads to effects that are observable (or conspicuously missing) in the mass spectrum.

3.1 Flavor symmetries and currents

Noether theorem. The Noether theorem states that any continuous symmetry transformation that leaves the classical action invariant implies the existence of a conserved current, where the corresponding charge is a constant of motion. This is true for spacetime symmetries and, in our context, global symmetries (but effectively also for local ones in the sense that each local symmetry has an underlying global symmetry). Let us exemplify the statement for a generic field theory with action $S = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu\phi_i)$. Consider a global transformation

$$\phi_i' = D_{ij}(\varepsilon)\,\phi_j = \left(e^{i\sum_a \varepsilon_a \mathsf{t}_a}\right)_{ij}\phi_j = \phi_i + \delta\phi_i \tag{3.1.1}$$

of the fields under some Lie group G, where ε_a are the group parameters, the t_a with $[\mathsf{t}_a, \mathsf{t}_b] = i f_{abc}\,\mathsf{t}_c$ are the generators of the Lie algebra in the representation to which the ϕ_i belong, and $D(\varepsilon)$ are the representation matrices. Compute the variation of the action with respect to the group parameter ε_a inside a spacetime volume V and for solutions of the classical equations of motion:

$$\begin{aligned}\delta S &= \int_V d^4x\,\delta\mathcal{L} = \int_V d^4x \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\,\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta(\partial_\mu\phi_i)\right] \\ &= \int_V d^4x \left[\partial_\mu\underbrace{\left(\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\delta\phi_i\right)}_{=:\,-\sum_a \varepsilon_a\, j_a^\mu} + \sum_i \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)}_{=\,0 \text{ for classical solutions}}\delta\phi_i\right].\end{aligned} \tag{3.1.2}$$

Here we considered a fixed volume V where the fields do not vanish on the surface, so the surface term does not vanish automatically. The second bracket vanishes for solutions of the Euler-Lagrange equations. Hence, if the classical action is invariant under the symmetry and thus $\delta S = 0$, there is one **conserved current** for each generator of the symmetry group when evaluated along the classical trajectories:

$$\partial_\mu \, j^\mu_a = 0 \qquad \text{with} \qquad j^\mu_a = -i \sum_{ij} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \, (\mathsf{t}_a)_{ij} \, \phi_j \,. \tag{3.1.3}$$

We can further use Gauss' law to convert the volume to a surface integral. For two spacelike hypersurfaces σ_1 and σ_2, if the surface term at spatial infinity is zero, this yields

$$\int\limits_V d^4x \, \partial_\mu \, j^\mu_a = \int\limits_{\partial V} d\sigma_\mu \, j^\mu_\alpha = \left[\int\limits_{\sigma_2} - \int\limits_{\sigma_1} \right] d\sigma_\mu \, j^\mu_\alpha = 0 \,. \tag{3.1.4}$$

In particular for two spacelike surfaces at fixed time t, the integral over a surface in four dimensions becomes a three-dimensional volume integral, where the four-vector $d\sigma_\mu = (d^3x, \mathbf{0})$ points in the time direction, and therefore we find a **conserved charge** for each generator t_a of the symmetry group:

$$Q_a(t) = \int d^3x \, j^0_a(x) = \text{const.} \tag{3.1.5}$$

Note that the currents and charges are still well-defined if the fields do not obey the classical equations of motion (then the second parenthesis in Eq. (3.1.2) is nonzero) or if the symmetry is classically broken (then the action is not invariant, $\delta S \neq 0$). In these cases the currents and charges are not conserved:

$$\partial_\mu \, j^\mu_a \neq 0 \,, \qquad \frac{d}{dt} \, Q_a(t) \neq 0 \,. \tag{3.1.6}$$

Quantization. When the classical field theory is quantized, the fields $\phi_i(x)$, currents $j^\mu_a(x)$ and charges $Q_a(t)$ become operators on the state space of the theory. As we will see later, the (anti-) commutation relations of the fields imply that the charges satisfy the same commutator relations as the generators of the symmetry group,

$$[Q_a, Q_b] = i f_{abc} \, Q_c \,, \tag{3.1.7}$$

so they form a representation of the Lie algebra on the state space (the **charge algebra**). This relation remains intact even if the charges are time-dependent, i.e., if the symmetry is broken. The charges can be used to construct a representation of the group on the state space under which the field operators transform,

$$U = e^{i \sum_a \varepsilon_a \, Q_a} \,, \qquad |\lambda'\rangle = U \, |\lambda\rangle \,, \qquad U \, \phi_i \, U^{-1} = D^{-1}_{ij} \, \phi_j \,, \tag{3.1.8}$$

which implements the classical relation (3.1.1) at the level of expectation values:

$$\langle \lambda_1' | \, \phi_i \, | \lambda_2' \rangle = \langle \lambda_1 | \, U^{-1} \, \phi_i \, U \, | \lambda_2 \rangle = D_{ij} \, \langle \lambda_1 | \, \phi_j \, | \lambda_2 \rangle \,. \tag{3.1.9}$$

(Note that later we will not always be consistent in the notation and denote the transformation matrices of the classical fields by U instead of D while leaving the operators $\exp(i\sum_a \varepsilon_a Q_a)$ unnamed.) By expanding $U \approx 1 + i\sum_a \varepsilon_a Q_a$ and $D \approx 1 + i\sum_a \varepsilon_a \mathsf{t}_a$, Eq. (3.1.8) entails

$$[Q_a, \phi_i] = -(\mathsf{t}_a)_{ij}\,\phi_j\,. \tag{3.1.10}$$

If the symmetry leaves the vacuum invariant, $U|0\rangle = |0\rangle$, then all generators Q_a must annihilate the vacuum: $Q_a|0\rangle = 0$, and we find

$$\langle 0|\phi_i|0\rangle = D_{ij}\,\langle 0|\phi_j|0\rangle = \langle 0|\phi_i|0\rangle + i\sum_a \varepsilon_a\,(\mathsf{t}_a)_{ij}\,\langle 0|\phi_j|0\rangle\,. \tag{3.1.11}$$

Thus, the vacuum expectation values must vanish for those directions ε_a that do not leave the ϕ_i invariant, which is the usual **'Wigner-Weyl realization'** of a symmetry: $\langle 0|\phi_i|0\rangle = 0$. Later we will study examples where the ϕ_i can be composite fields (such as $\bar{\psi}\psi$) or also collections of different fields (e.g. σ and π_a in the σ model).

The **Heisenberg equations of motion**, on the other hand, are a consequence of translation invariance $\partial_\mu F(\phi) = i[P_\mu, F(\phi)]$, which holds for arbitrary polynomials of the fields including the charges $Q_a(t)$:

$$\frac{dQ_a}{dt} = i\,[H_{\text{QCD}}, Q_a]\,. \tag{3.1.12}$$

Therefore, if the charges are conserved, they commute with the Hamiltonian and have a common eigenvalue spectrum. In other words, the mass spectrum of the theory can be labeled by the irreducible representations of the symmetry group, which will lead to the flavor multiplets of hadrons.

In addition to the explicit breaking of a symmetry, there are also other possibilities how classical symmetries can be broken at the quantum level:

- **Spontaneous symmetry breaking:** Here the classical action is still invariant under the global symmetry and the currents are conserved, $\partial_\mu\, j^\mu_a = 0$, but the vacuum and the correlation functions of the theory lose this symmetry and develop condensates $\langle 0|\phi_i|0\rangle \neq 0$. As a consequence, $U|0\rangle \neq |0\rangle$ and there are charges which do not annihilate the vacuum: $Q_a|0\rangle \neq 0$ (we will refine these statements in Sec. 4.2). For each such charge there is a **massless Goldstone boson**. Spontaneous symmetry breaking is a dynamical effect due to the dynamics inherent in the theory, so one may as well turn the argument around and argue that it happens *because* the dynamics contains massless long-range interactions. The QCD example is chiral symmetry or, more precisely, the group $SU(N_f)_A$ for vanishing quark masses.

- **Anomalous symmetry breaking:** Also here the classical action is invariant, but the symmetry is broken at the quantum level due to regularization, i.e. if there is no regulator that preserves the classical symmetry. We already mentioned the anomalous breaking of scale invariance; other typical candidates are again axial symmetries: in dimensional regularization, γ_5 has no natural extension to $d \neq 4$ dimensions; a Pauli-Villars regulator breaks chiral symmetry explicitly due to a mass term, etc. In Sec. 4.3 we will study the $U(1)_A$ anomaly in QCD. In contrast to spontaneous symmetry breaking, also the currents are no longer conserved and pick up additional terms so that $\partial_\mu\, j^\mu_a \neq 0$.

3.1.1 Symmetries of the QCD Lagrangian

In order to discuss QCD's flavor symmetries, we only need to consider the quark parts of the QCD Lagrangian since only the quark fields carry flavor labels:

$$\mathcal{L} = \overline{\psi}\,(i\partial\!\!\!/ - \mathsf{M})\,\psi + g\,\overline{\psi}\,A\!\!\!/\,\psi \qquad \text{with} \qquad \psi_{\alpha,i}(x)\,, \quad \overline{\psi}_{\alpha,i}(x)\,. \tag{3.1.13}$$

In the following the index $i = 1 \dots N_f$ denotes the flavor and we suppress the color indices. For simplicity we also work with unrenormalized quantities and discuss renormalization when necessary. The spinor fields $\overline{\psi}_{\alpha,i}(x)$, $\psi_{\alpha,i}(x)$ transform under the fundamental representation of $SU(N_f)$:

$$\psi'(x) = U\psi(x), \qquad \overline{\psi}'(x) = \overline{\psi}(x)\,U^\dagger \qquad \text{with} \quad U = e^{i\sum_a \varepsilon_a \mathsf{t}_a}\,. \tag{3.1.14}$$

The t_a are the $SU(N_f)$ generators, e.g., the Pauli matrices $\mathsf{t}_a = \tau_a/2$ for two flavors and Gell-Mann matrices $\mathsf{t}_a = \lambda_a/2$ for three flavors (see Appendix A). In the two-flavor case, the diagonal quark mass matrix in the Lagrangian has the form

$$\mathsf{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{m_u + m_d}{2}\,\mathbb{1} + (m_u - m_d)\,\mathsf{t}_3\,, \tag{3.1.15}$$

whereas in the three-flavor case it is given by $\mathsf{M} = \mathrm{diag}(m_u, m_d, m_s)$ or

$$\mathsf{M} = \frac{m_u + m_d + m_s}{3}\,\mathbb{1} + (m_u - m_d)\,\mathsf{t}_3 + \frac{m_u + m_d - 2m_s}{\sqrt{3}}\,\mathsf{t}_8\,. \tag{3.1.16}$$

Flavor symmetries. Consider the following global transformations of the quark fields:

$$\begin{aligned} e^{i\sum_a \varepsilon_a \mathsf{t}_a} &\in SU(N_f)_V\,, \qquad & e^{i\varepsilon} &\in U(1)_V\,, \\ e^{i\gamma_5\sum_a \varepsilon_a \mathsf{t}_a} &\in SU(N_f)_A\,, \qquad & e^{i\gamma_5\varepsilon} &\in U(1)_A\,, \end{aligned} \tag{3.1.17}$$

where $SU(N_f)_V$ denotes the usual transformation from Eq. (3.1.14). The subscripts V and A indicate that these transformations will induce vector and axialvector currents. The axial transformations involve γ_5 matrices and in the $U(1)$ cases ε is just a number. The infinitesimal transformations of the quark and antiquark fields read

$$SU(N_f)_V: \qquad \delta\psi = \sum_a \varepsilon_a\,\mathsf{t}_a\,i\psi\,, \qquad \delta\overline{\psi} = -i\overline{\psi}\,\sum_a \varepsilon_a\,\mathsf{t}_a\,, \tag{3.1.18}$$

$$SU(N_f)_A: \qquad \delta\psi = \gamma_5\sum_a \varepsilon_a\,\mathsf{t}_a\,i\psi\,, \quad \delta\overline{\psi} = i\overline{\psi}\,\gamma_5\,\sum_a \varepsilon_a\mathsf{t}_a\,. \tag{3.1.19}$$

$$U(1)_V: \qquad \delta\psi = \varepsilon\,i\psi\,, \qquad \delta\overline{\psi} = -i\overline{\psi}\,\varepsilon\,, \tag{3.1.20}$$

$$U(1)_A: \qquad \delta\psi = \varepsilon\,\gamma_5\,i\psi\,, \qquad \delta\overline{\psi} = i\overline{\psi}\gamma_5\,\varepsilon\,. \tag{3.1.21}$$

Note the positive signs for the $\delta\overline{\psi}$ terms in the axial cases, which follow from the anticommutation of γ_5 and γ_0 in obtaining $\overline{\psi} = \psi^\dagger\gamma_0$:

$$(\gamma_5\,\mathsf{t}_a\,i\psi)^\dagger\,\gamma_0 = -i\psi^\dagger\,\mathsf{t}_a\,\gamma_5\,\gamma_0 = +i\overline{\psi}\,\mathsf{t}_a\,\gamma_5\,. \tag{3.1.22}$$

We will make frequent use of the following **quark bilinears**:

$$j_a^\Gamma(x) := \overline{\psi}(x)\,\Gamma\,\mathsf{t}_a\,\psi(x)\,, \qquad j^\Gamma(x) := \overline{\psi}(x)\,\Gamma\,\psi(x)\,, \tag{3.1.23}$$

where $\Gamma \in \{\gamma^\mu, \gamma^\mu\gamma_5, \mathbb{1}, i\gamma_5\}$ are vector, axialvector, scalar and pseudoscalar Dirac matrices. We denote the corresponding vector, axialvector, scalar and pseudoscalar currents or densities $j^\Gamma_{(a)}(x)$ by [1]

$$\gamma^\mu \to V^\mu_{(a)}(x), \qquad \gamma^\mu\gamma_5 \to A^\mu_{(a)}(x), \qquad \mathbb{1} \to S_{(a)}(x), \qquad i\gamma_5 \to P_{(a)}(x). \tag{3.1.24}$$

These quantities are all hermitian, e.g.

$$P^\dagger = (\overline{\psi}\,i\gamma_5\psi)^\dagger = -i\psi^\dagger\gamma_5\,\gamma_0\,\psi = +i\overline{\psi}\,\gamma_5\,\psi = P\,. \tag{3.1.25}$$

In the following we investigate the symmetry transformations $U(1)_V \times SU(N_f)_V \times SU(N_f)_A \times U(1)_A$ in detail.

■ $\mathbf{U(1)_V}$: The action is invariant under a global phase transformation $\psi' = e^{i\varepsilon}\psi$. The corresponding flavor-singlet vector current according to Eq. (3.1.2) is

$$V^\mu = -\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_{\alpha,i})}\,(i\psi_{\alpha,i}) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}_{\alpha,i})}\,(-i\overline{\psi}_{\alpha,i})\right] = \overline{\psi}\,\gamma^\mu\,\psi\,, \tag{3.1.26}$$

where we used (cf. Eq. (2.1.40))

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\overline{\psi}\,\gamma^\mu\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0\,. \tag{3.1.27}$$

Current conservation $\partial_\mu V^\mu = 0$ can be verified by inserting the solutions of the classical Dirac equations of motion from Eq. (2.1.41), where A^μ is the gluon field,

$$\partial\!\!\!/\,\psi = (g\,A\!\!\!/ - \mathsf{M})\,i\psi\,, \qquad \overline{\psi}\,\overleftarrow{\partial\!\!\!/} = -i\overline{\psi}\,(g\,A\!\!\!/ - \mathsf{M})\,, \tag{3.1.28}$$

and thus $\partial_\mu V^\mu = \overline{\psi}\,\partial\!\!\!/\,\psi + \overline{\psi}\,\overleftarrow{\partial\!\!\!/}\,\psi = 0$. The conserved charge is

$$Q^V(t) = \int d^3x\,\overline{\psi}\,\gamma^0\,\psi = \int d^3x\,\psi^\dagger\,\psi = \text{const.} \tag{3.1.29}$$

and reflects fermion number conservation, because in the quantum field theory it counts the number of quarks minus antiquarks in the state. If we define $n_q = (\#q) - (\#\bar{q})$ for each flavor, then the eigenvalue of Q^V (which we also call Q^V) is the baryon number. For three flavors:

$$B = \frac{Q^V}{3} = \frac{n_u + n_d + n_s}{3}\,, \tag{3.1.30}$$

and the $U(1)_V$ symmetry entails **baryon number conservation**.

[1] Here is a clash of notation: A^μ denotes both the axialvector current and the gluon field. Fortunately we won't be dealing with gluons for a while, and if so we will use the gluon field-strength tensor $F^{\mu\nu}$ instead. Unless stated otherwise, from now on A^μ will refer to an axialvector current.

■ **$\mathbf{SU(N_f)_V}$:** is explicitly broken by the mass matrix $\mathsf{M} \neq m\,\mathbb{1}$ since $U^\dagger \mathsf{M}\, U \neq \mathsf{M}$. We can still write down the currents, one for each generator of the group, and compute their divergences:

$$V_a^\mu = \bar{\psi}\,\gamma^\mu\,\mathsf{t}_a\,\psi\,, \qquad \partial_\mu\,V_a^\mu = i\bar{\psi}\,[\mathsf{M},\mathsf{t}_a]\,\psi\,. \tag{3.1.31}$$

The action is only invariant if all quark masses are identical. In that case the $(N_f^2 - 1)$ vector currents are conserved, $\partial_\mu\,V_a^\mu = 0$, and so are the corresponding charges:

$$Q_a^V(t) = \int d^3x\,\psi^\dagger\,\mathsf{t}_a\,\psi = \text{const.} \tag{3.1.32}$$

Because the diagonal generators (t_3 in the two-flavor and t_3, t_8 in the three-flavor case) commute with each other and hence also with the mass matrix M, the corresponding **isospin** and **hypercharge** currents

$$\begin{aligned} V_3^\mu &= \bar{\psi}\,\gamma^\mu\,\mathsf{t}_3\,\psi = \frac{1}{2}\left(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d\right), \\ V_8^\mu &= \bar{\psi}\,\gamma^\mu\,\mathsf{t}_8\,\psi = \frac{1}{2\sqrt{3}}\left(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d - 2\bar{s}\gamma^\mu s\right) \end{aligned} \tag{3.1.33}$$

are *always* conserved, even if $\mathsf{M} \neq m\,\mathbb{1}$. In combination with the vector-singlet current $V^\mu = \bar{\psi}\gamma^\mu\psi$ from Eq. (3.1.26), this implies that the flavor-diagonal vector currents $\bar{u}\gamma^\mu u$, $\bar{d}\,\gamma^\mu d$ and $\bar{s}\,\gamma^\mu s$ are individually conserved, which reflects flavor conservation in QCD. The corresponding charges are the third component of the isospin I_3 and the hypercharge Y:

$$I_3 = Q_3^V = \frac{n_u - n_d}{2}\,, \qquad Y = \frac{2}{\sqrt{3}}\,Q_8^V = \frac{n_u + n_d - 2n_s}{3}\,. \tag{3.1.34}$$

This is what allows us to arrange hadrons in $\{I_3, Y\}$ multiplets even if the underlying flavor symmetry is broken due to the unequal quark masses. From the eigenvalues of B, I_3 and Y we obtain

$$Y = B + S\,, \qquad Q = I_3 + \frac{Y}{2} = \frac{2}{3}\,n_u - \frac{1}{3}\,n_d - \frac{1}{3}\,n_s\,, \tag{3.1.35}$$

where $S = -n_s$ is the strangeness and Q the electric charge of the state. The relation $Q = I_3 + Y/2$ is the **Gell-Mann-Nishijima formula**.

The remaining flavor-changing vector currents have divergences proportional to quark-mass differences; if we go back to the two-flavor case with $m_u \neq m_d$ and use instead of $\mathsf{t}_{1,2} = \tau_{1,2}/2$ the generators

$$\mathsf{t}_+ = \mathsf{t}_1 + i\mathsf{t}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{t}_- = \mathsf{t}_1 - i\mathsf{t}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.1.36}$$

we obtain

$$\partial_\mu V_\pm^\mu = i\bar{\psi}\,[\mathsf{M},\mathsf{t}_\pm]\,\psi = i(m_u - m_d)\begin{cases} \bar{u}d \\ -\bar{d}u\,. \end{cases} \tag{3.1.37}$$

- $\mathbf{SU(N_f)_A}$: is explicitly broken by the mass matrix $\mathsf{M} \neq 0$:

$$A_a^\mu = \overline{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi\,, \qquad \partial_\mu A_a^\mu = i\overline{\psi}\,\{\mathsf{M},\mathsf{t}_a\}\,\gamma_5\,\psi\,. \tag{3.1.38}$$

Even if all quark masses are equal, there remains a non-zero contribution proportional to the quark mass:

$$\partial_\mu A_a^\mu = 2m\,\overline{\psi}\,i\gamma_5\,\mathsf{t}_a\psi = 2mP_a\,. \tag{3.1.39}$$

This is the **PCAC relation** ('partially conserved axialvector current'): the divergence of the axialvector current is proportional to a pseudoscalar density. This equation will become extremely useful later. Using (3.1.36) in the two-flavor case, we obtain

$$\begin{aligned} \partial_\mu A_+^\mu &= i(m_u+m_d)\,\bar{u}\gamma_5 d\,,\\ \partial_\mu A_-^\mu &= i(m_u+m_d)\,\bar{d}\gamma_5 u\,,\\ \partial_\mu A_3^\mu &= im_u\,\bar{u}\gamma_5 u - im_d\,\bar{d}\gamma_5 d\,, \end{aligned} \tag{3.1.40}$$

which are the creation operators for the three pions π^+, π^- and π^0.

On the other hand, in the **chiral limit** where $\mathsf{M}=0$, Eq. (3.1.38) entails that the axial currents and the corresponding axial charges are conserved:

$$\partial_\mu A_a^\mu = 0 \quad \Rightarrow \quad Q_a^A(t) = \int d^3x\,\psi^\dagger\gamma_5\,\mathsf{t}_a\,\psi = \text{const.} \tag{3.1.41}$$

Since the vector currents are conserved as well in that case, we have an enlarged flavor symmetry, namely **chiral symmetry**: $SU(N_f)_V \times SU(N_f)_A \simeq SU(N_f)_L \times SU(N_f)_R$. Later we will see that the $SU(N_f)_A$ part is spontaneously broken at the quantum level; nevertheless all relations for the currents remain valid.

Because of the spontaneous breaking of the axial part, in QCD the V/A terminology is more useful than the L/R notation — in contrast to the electroweak theory, where left- and right-handed fermions enter asymmetrically in the Lagrangian. Nevertheless, let us collect some relations that will become useful later. We define the chiral projectors

$$\mathsf{P}_\pm := \frac{1}{2}\,(\mathbb{1}\pm\gamma_5) \quad \Rightarrow \quad \mathsf{P}_\omega = \mathsf{P}_\omega^\dagger, \quad \sum_\omega \mathsf{P}_\omega = \mathbb{1}, \quad \mathsf{P}_\omega\,\mathsf{P}_{\omega'} = \delta_{\omega\omega'}\,\mathsf{P}_\omega\,, \tag{3.1.42}$$

where chirality is denoted by the index $\omega=+$ (R, right-handed) or $\omega=-$ (L, left-handed). The projectors can be used to define right- and left-handed spinors:

$$\psi_\omega = \mathsf{P}_\omega\,\psi\,, \qquad \overline{\psi}_\omega = (\mathsf{P}_\omega\,\psi)^\dagger\gamma_0 = \psi^\dagger\,\mathsf{P}_\omega\,\gamma_0 = \overline{\psi}\,\mathsf{P}_{-\omega}\,, \qquad \psi = \sum_\omega \psi_\omega\,. \tag{3.1.43}$$

Now consider the product of infinitesimal vector and axialvector transformations:

$$U_V\,U_A = e^{i\varepsilon_V}\,e^{i\varepsilon_A\gamma_5} = 1 + i\varepsilon_V + i\varepsilon_A\gamma_5 + \dots = 1 + i\sum_\omega \varepsilon_\omega\,\mathsf{P}_\omega = \sum_\omega U_\omega\,\mathsf{P}_\omega\,. \tag{3.1.44}$$

Here we abbreviated $\varepsilon_{V,A} = \sum_a \varepsilon_a^{V,A}\,\mathsf{t}_a$ and defined $\varepsilon_\pm = \varepsilon_V \pm \varepsilon_A$ and $U_\omega = e^{i\varepsilon_\omega}$, which are all just flavor matrices. As a consequence, the left- and right-handed spinors transform as

$$\psi'_\omega = \mathsf{P}_\omega\,\psi' = \mathsf{P}_\omega\,U_V\,U_A\,\psi = \mathsf{P}_\omega\sum_{\omega'} U_{\omega'}\,\mathsf{P}_{\omega'}\,\psi = U_\omega\,\psi_\omega\,. \tag{3.1.45}$$

Therefore, they transform *separately* under $SU(N_f)_L \times SU(N_f)_R$, with independent group parameters ε_a^R and ε_a^L:

$$\psi'_\omega = U_\omega\,\psi_\omega\,, \qquad \overline{\psi}'_\omega = \overline{\psi}_\omega\,U_\omega^\dagger\,, \qquad U_\omega = e^{i\sum_a \varepsilon_a^\omega\,\mathsf{t}_a}\,, \qquad U_\omega^\dagger = U_\omega^{-1}\,. \tag{3.1.46}$$

Now let us cast the currents $\bar{\psi}\,\Gamma\,\psi$ with $\Gamma \in \{\gamma^\mu,\,\gamma^\mu\gamma_5,\,\mathbb{1},\,\gamma_5,\,\sigma^{\mu\nu}\}$ in the L/R notation:

$$\bar{\psi}\,\Gamma\,\psi = \sum_\omega \bar{\psi}\,\Gamma\,\mathsf{P}_\omega^2\,\psi = \begin{cases} \displaystyle\sum_\omega \bar{\psi}\,\mathsf{P}_{-\omega}\,\Gamma\,\mathsf{P}_\omega\,\psi = \sum_\omega \bar{\psi}_\omega\,\Gamma\,\psi_\omega & \dots\Gamma \in \{\gamma^\mu,\,\gamma^\mu\gamma_5\}\,, \\ \displaystyle\sum_\omega \bar{\psi}\,\mathsf{P}_\omega\,\Gamma\,\mathsf{P}_\omega\,\psi = \sum_\omega \bar{\psi}_{-\omega}\,\Gamma\,\psi_\omega & \dots\Gamma \in \{\mathbb{1},\,\gamma_5,\,\sigma^{\mu\nu}\}. \end{cases} \tag{3.1.47}$$

This means that for the currents constructed from the Dirac matrices γ^μ and $\gamma^\mu\gamma_5$ only the diagonal terms survive ($LL+RR$), whereas for the remaining ones only the mixed terms survive ($LR+RL$). How do these transform under chiral symmetry? The diagonal ones are invariant,

$$\sum_\omega \bar{\psi}'_\omega\,\Gamma\,\psi'_\omega = \sum_\omega \bar{\psi}_\omega\,U_\omega^\dagger\,\Gamma\,U_\omega\,\psi_\omega = \sum_\omega \bar{\psi}_\omega\,\Gamma\,\psi_\omega\,, \tag{3.1.48}$$

because U_ω is just a flavor matrix and $U_\omega^\dagger\,U_\omega = 1$. The off-diagonal currents, on the other hand, are not invariant because $U_{-\omega}^\dagger\,U_\omega \neq 1$:

$$\sum_\omega \bar{\psi}'_{-\omega}\,\Gamma\,\psi'_\omega = \sum_\omega \bar{\psi}_\omega\,\Gamma\,U_{-\omega}^\dagger\,U_\omega\,\psi_\omega \neq \sum_\omega \bar{\psi}_{-\omega}\,\Gamma\,\psi_\omega\,. \tag{3.1.49}$$

As a consequence, the massless Lagrangian $\bar{\psi}\,i\partial\!\!\!/\,\psi$ is chirally invariant, whereas a mass term $\bar{\psi}\psi$ breaks chiral symmetry since it mixes left- and right-handed components. The Lagrangian (3.1.13) takes the form

$$\mathcal{L} = \sum_\omega \left(\bar{\psi}_\omega\,i\partial\!\!\!/\,\psi_\omega - \bar{\psi}_{-\omega}\,\mathsf{M}\,\psi_\omega\right). \tag{3.1.50}$$

From the global $SU(N_f)\times SU(N_f)$ transformations we can define $2\times(N_f^2-1)$ currents and charges, which are only conserved in the chiral limit:

$$j^\mu_{a,\omega} = \bar{\psi}_\omega\gamma^\mu\,\mathsf{t}_a\,\psi_\omega\,, \qquad Q_{a,\omega} = \int d^3x\,\psi_\omega^\dagger\,\mathsf{t}_a\,\psi_\omega\,. \tag{3.1.51}$$

Inserting the Dirac equations for ψ_ω and $\bar{\psi}_\omega$, their divergences for $\mathsf{M}\neq 0$ become

$$\partial_\mu\,j^\mu_{a,\omega} = i\left(\bar{\psi}_{-\omega}\,\mathsf{M}\,\mathsf{t}_a\,\psi_\omega - \bar{\psi}_\omega\,\mathsf{t}_a\,\mathsf{M}\,\psi_{-\omega}\right). \tag{3.1.52}$$

The vector and axialvector currents from Eqs. (3.1.31) and (3.1.38) and corresponding charges are linear combinations of these, with $V = R+L$ and $A = R-L$:

$$\begin{aligned} V^\mu_a &= \bar{\psi}\,\gamma^\mu\,\mathsf{t}_a\,\psi = \sum_\omega \bar{\psi}_\omega\gamma^\mu\,\mathsf{t}_a\,\psi_\omega = \sum_\omega j^\mu_{a,\omega}\,, \\ A^\mu_a &= \bar{\psi}\,\gamma^\mu\gamma_5\,\mathsf{t}_a\,\psi = \sum_\omega \bar{\psi}_\omega\gamma^\mu\underbrace{\gamma_5}_{\mathsf{P}_+-\mathsf{P}_-}\mathsf{t}_a\,\psi_\omega = j^\mu_{a,+} - j^\mu_{a,-}\,. \end{aligned} \tag{3.1.53}$$

■ **$\mathbf{U(1)_A}$** : is classically conserved for $\mathsf{M}=0$, but not preserved after quantization which leads to the $U(1)_A$ anomaly. The divergence of the axialvector singlet current picks up an anomalous contribution whose origin and consequences we will discuss in Sec. 4.3:

$$A^\mu = \bar{\psi}\,\gamma^\mu\gamma_5\,\psi\,, \qquad \partial_\mu A^\mu = 2i\,\bar{\psi}\,\mathsf{M}\,\gamma_5\,\psi + \frac{g^2 N_f}{(4\pi)^2}\,\widetilde{F}^{\mu\nu}_a\,F^a_{\mu\nu}\,. \tag{3.1.54}$$

3.1.2 Symmetry relations at the quantum level

Current and charge algebra. The symmetry relations we have discussed so far hold for the classical currents and charges. When we quantize the theory, the quark fields become operators on the state space which satisfy the anticommutation relations (2.2.67):

$$\begin{aligned} \left\{\psi_{\alpha i}(x), \psi_{\beta j}^\dagger(y)\right\}_{x^0=y^0} &= \delta^3(\boldsymbol{x}-\boldsymbol{y})\, \delta_{\alpha\beta}\, \delta_{ij}\,, \\ \left\{\psi_{\alpha i}(x), \psi_{\beta j}(y)\right\}_{x^0=y^0} &= \left\{\psi_{\alpha i}^\dagger(x), \psi_{\beta j}^\dagger(y)\right\}_{x^0=y^0} = 0\,. \end{aligned} \tag{3.1.55}$$

As a consequence, also the currents in Eq. (3.1.23) become operators,

$$j_a^\Gamma(x) = \overline{\psi}(x)\, \Gamma\, \mathsf{t}_a\, \psi(x)\,, \qquad j^\Gamma(x) = \overline{\psi}(x)\, \Gamma\, \psi(x)\,, \tag{3.1.56}$$

which satisfy the equal-time commutation relations

$$\begin{aligned} \left[j_a^\Gamma(x), j_b^{\Gamma'}(y)\right]_{x^0=y^0} &= \left[i f_{abc}\, j_c^{\Gamma_+}(x) + d_{abc}\, j_c^{\Gamma_-}(x) + \frac{\delta_{ab}}{N}\, j^{\Gamma_-}(x)\right] \delta^3(\boldsymbol{x}-\boldsymbol{y}), \\ \left[j_a^\Gamma(x), j^{\Gamma'}(y)\right]_{x^0=y^0} &= 2 j_a^{\Gamma_-}(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) \end{aligned} \tag{3.1.57}$$

with $\Gamma_\pm = \frac{1}{2}\,(\Gamma\gamma^0\,\Gamma' \pm \Gamma'\gamma^0\,\Gamma)$. These relations are valid independently of whether the currents are conserved or not. Moreover, for spacelike distances the commutators vanish, which ensures causality from Eq. (2.2.1):

$$\left[j_a^\Gamma(x), j_b^{\Gamma'}(y)\right] = \left[j_a^\Gamma(x), j^{\Gamma'}(y)\right] = 0 \qquad \text{for} \qquad (x-y)^2 < 0\,. \tag{3.1.58}$$

The proof is straightforward. We write

$$\left[j_a^\Gamma(x), j_b^{\Gamma'}(y)\right]_{x^0=y^0} = (\gamma_0\Gamma)_{\alpha\beta}\, (\mathsf{t}_a)_{ij}\, (\gamma_0\Gamma')_{\gamma\delta}\, (\mathsf{t}_b)_{kl} \left[\psi_{\alpha i}^\dagger(x)\, \psi_{\beta j}(x), \psi_{\gamma k}^\dagger(y)\, \psi_{\delta l}(y)\right]_{x^0=y^0} \tag{3.1.59}$$

and use the identity

$$[AB, CD] = A\,\{B,C\}\,D - C\,\{A,D\}\,B - \frac{\{A,C\}\,[B,D] + [A,C]\,\{B,D\}}{2} \tag{3.1.60}$$

together with the anticommutation relations (3.1.55) for the quark fields. The terms with $\{A,C\}$ and $\{B,D\}$ vanish and the commutator on the r.h.s. of (3.1.59) becomes

$$[\dots]_{x^0=y^0} = \delta^3(\boldsymbol{x}-\boldsymbol{y}) \left[\delta_{\beta\gamma}\, \delta_{jk}\, \psi_{\alpha i}^\dagger(x)\, \psi_{\delta l}(y) - \delta_{\alpha\delta}\, \delta_{il}\, \psi_{\gamma k}^\dagger(y)\, \psi_{\beta j}(x)\right]_{x^0=y^0}\,. \tag{3.1.61}$$

Since $\boldsymbol{x}=\boldsymbol{y}$ and $x^0=y^0$ entails $x=y$, then in combination with the Dirac and flavor matrices the full commutator is

$$\left[j_a^\Gamma(x), j_b^{\Gamma'}(y)\right]_{x^0=y^0} = \delta^3(\boldsymbol{x}-\boldsymbol{y})\, \overline{\psi}(x) \left(\Gamma\gamma_0\,\Gamma'\, \mathsf{t}_a\, \mathsf{t}_b - \Gamma'\gamma_0\,\Gamma\, \mathsf{t}_b\, \mathsf{t}_a\right) \psi(x)\,. \tag{3.1.62}$$

With

$$AX - BY = \frac{A+B}{2}\,(X-Y) + \frac{A-B}{2}\,(X+Y)\,, \qquad \Gamma_\pm = \frac{\Gamma\gamma^0\,\Gamma' \pm \Gamma'\gamma^0\,\Gamma}{2} \tag{3.1.63}$$

and the (anti-) commutation relations (A.1.2) and (A.1.7) for the $SU(N)$ generators

$$[\mathsf{t}_a, \mathsf{t}_b] = i f_{abc}\, \mathsf{t}_c\,, \qquad \{\mathsf{t}_a, \mathsf{t}_b\} = \frac{1}{N}\, \delta_{ab} + d_{abc}\, \mathsf{t}_c \tag{3.1.64}$$

we arrive at the result in Eq. (3.1.57). We note that for commutators involving spatial components of the currents these relations must be taken with some caution because additional **Schwinger terms** may arise on the r.h.s., which are derivatives of $\delta-$functions of the form $\partial_i\, \delta^3(\boldsymbol{x}-\boldsymbol{y})$.

Some examples of (3.1.57) involving temporal current components are:

- For Γ, $\Gamma' \in \{\gamma^0, \gamma^0\gamma_5\}$ we find

$$\Gamma_+ = \begin{cases} \gamma^0 & \ldots \; \Gamma = \Gamma' \,, \\ \gamma^0\gamma_5 & \ldots \; \Gamma \neq \Gamma' \,, \end{cases} \qquad \Gamma_- = 0 \tag{3.1.65}$$

which leads to the so-called **'local current algebra'**:

$$\begin{aligned} [V_a^0(x), V_b^0(y)]_{x^0=y^0} &= i f_{abc}\, V_c^0(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \\ [V_a^0(x), A_b^0(y)]_{x^0=y^0} &= i f_{abc}\, A_c^0(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \\ [A_a^0(x), A_b^0(y)]_{x^0=y^0} &= i f_{abc}\, V_c^0(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \end{aligned} \tag{3.1.66}$$

The time components V_a^0 and A_a^0 form a closed algebra since they obey equal-time commutation relations with the structure constants of the Lie algebra, and the Dirac $\delta-$functions additionally ensure that all commutators vanish for $x \neq y$. If we further integrate over $\int d^3x$ and $\int d^3y$, we obtain the corresponding **charge algebra**:

$$[Q_a^V, Q_b^V] = [Q_a^A, Q_b^A] = i f_{abc}\, Q_c^V \,, \qquad [Q_a^V, Q_b^A] = i f_{abc}\, Q_c^A \,. \tag{3.1.67}$$

Therefore, the charges are the generators of the symmetry group when acting on the state space. Actually, because the $SU(N_f)_A$ symmetry is spontaneously broken, the axial charges are not well-defined in the chiral limit and it is more practical to work directly with the time components of the currents.

- For $\Gamma = \gamma^0\gamma_5$ and $\Gamma' = i\gamma_5$ we find $\Gamma_+ = 0$, $\Gamma_- = -i$ and therefore

$$\left[Q_a^A, P_b(x)\right] = -i\left[\frac{\delta_{ab}}{N_f}\, S(x) + d_{abc}\, S_c(x)\right], \tag{3.1.68}$$

where $S(x) = \bar\psi(x)\,\psi(x)$ is the scalar density. Its vacuum expectation value is the scalar quark condensate which turns out to be nonzero due to spontaneous chiral symmetry breaking, and later we will use this relation for proving Goldstone's theorem and deriving the Gell-Mann-Oakes-Renner relation.

- Using the relation $[AB, C] = A\,\{B, C\} - \{A, C\}\,B$, we can similarly obtain the commutation relations of the currents with the quark fields,

$$\begin{aligned} \left[\, j_a^\Gamma(x), \psi(y)\right]_{x^0=y^0} &= -\big(\mathsf{t}_a\, \gamma^0\, \Gamma\, \psi(x)\big)\, \delta^3(\boldsymbol{x}-\boldsymbol{y}), \\ \left[\, j_a^\Gamma(x), \bar\psi(y)\right]_{x^0=y^0} &= \big(\bar\psi(x)\, \Gamma\, \gamma^0\, \mathsf{t}_a\big)\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \end{aligned} \tag{3.1.69}$$

for example for the vector currents ($\Gamma = \gamma^0$):

$$\begin{aligned} \left[\, V_a^0(x), \psi(y)\right]_{x^0=y^0} &= -\mathsf{t}_a\, \psi(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y}), \\ \left[\, V_a^0(x), \bar\psi(y)\right]_{x^0=y^0} &= \bar\psi(x)\, \mathsf{t}_a\, \delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \end{aligned} \tag{3.1.70}$$

Integrating over $\int d^3x$, we get the commutation relations of the charges with the fields:

$$\left[\, Q_a^V(x_0), \psi(x)\right] = -\mathsf{t}_a\, \psi(x)\,, \qquad \left[\, Q_a^V(x_0), \bar\psi(x)\right] = \psi(x)\, \mathsf{t}_a\,. \tag{3.1.71}$$

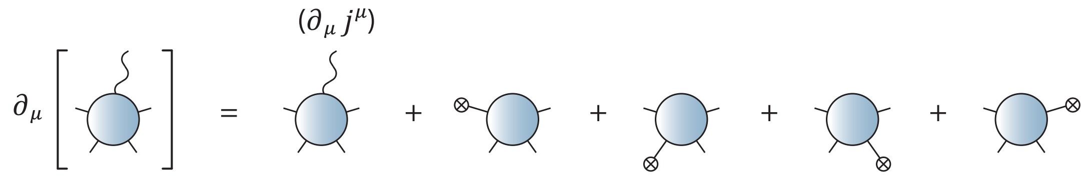

FIG. 3.1: Generic form of a Ward-Takahashi identity from Eq. (3.1.74).

Ward-Takahashi identities. Ultimately we would like to turn the classical symmetry relations into identities for the correlation functions of the QFT. These are the **Ward-Takahashi identities (WTIs)**, which relate the $n-$point and $(n+1)-$point functions of the theory with each other. They are usually derived in the path-integral approach (and we will come back to this below), but it is somewhat more transparent to work them out using canonical quantization.

Consider two generic field operators $j^\mu(x)$, $\varphi(y)$ at different spacetime points. The divergence of their time-ordered product with respect to x (with $\partial^x_\mu = \partial/\partial x^\mu$) is

$$\begin{aligned}\partial^x_\mu \left(\mathsf{T}\, j^\mu(x)\,\varphi(y)\right) &= \partial^x_\mu \left[\Theta(x^0-y^0)\, j^\mu(x)\,\varphi(y) + \Theta(y^0-x^0)\,\varphi(y)\, j^\mu(x)\right] \\ &= \mathsf{T}\left(\partial_\mu j^\mu(x)\right)\varphi(y) + \delta(x^0-y^0)\left[j^0(x),\varphi(y)\right].\end{aligned} \tag{3.1.72}$$

The first term comes from the derivative of $j^\mu(x)$ (simply resum the time orderings) and the second one results from differentiating the step functions:

$$\partial^x_\mu\,\Theta(x^0-y^0) = -\partial^x_\mu\,\Theta(y^0-x^0) = \delta(x^0-y^0)\,\delta_{0\mu}\,. \tag{3.1.73}$$

Eq. (3.1.72) is quite general and retains its structure for products of n different fields (which can also be fermionic). In the general case one has to write down all possible time orderings; the time-ordering of $n+1$ distinct space-time points leads to $(n+1)!$ terms, each of which includes products of n step functions. If fermion fields are involved, the individual time-ordered terms pick up minus signs arising from the anticommutativity. In either case, the final result is the same:

$$\begin{aligned}\partial^x_\mu \left(\mathsf{T}\, j^\mu(x)\,\varphi_1(x_1)\dots\varphi_n(x_n)\right) &= \mathsf{T}\left(\partial_\mu j^\mu(x)\right)\varphi_1(x_1)\dots\varphi_n(x_n) \\ &+ \sum_{k=1}^n \delta(x^0-x_k^0)\,\mathsf{T}\,\varphi_1(x_1)\dots\left[j^0(x),\varphi_k(x_k)\right]\dots\varphi_n(x_n)\,.\end{aligned} \tag{3.1.74}$$

If we take its vacuum expectation value $\langle 0|\,\dots\,|0\rangle$, it relates the $(n+1)-$point function, where one leg corresponds to the external current, to the $n-$point functions since the commutators in the second row are proportional to the fields, cf. Eq. (3.1.69). This is the generic form of a Ward-Takahashi identity and illustrated in Fig. 3.1. Current conservation (or its absence) only enters in the first term on the r.h.s., which vanishes if the current is conserved.

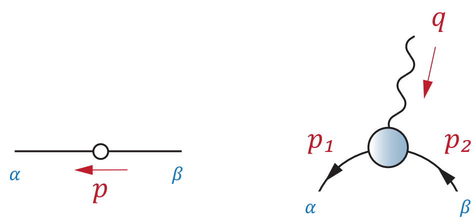

FIG. 3.2: Quark propagator and three-point function in Eqs. (3.1.75) and (3.1.76).

Let us apply this result to QCD. The quark propagator is the two-point function

$$S_{\alpha\beta}(x_1, x_2) = \langle 0|\,\mathsf{T}\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)\,|0\rangle\,. \tag{3.1.75}$$

How the quark couples to a vector, axialvector, scalar or pseudoscalar current (e.g. photons, $Z-$bosons, pions, ...) is encoded in the three-point function (see Fig. 3.2)

$$G^\Gamma_{a,\alpha\beta}(x, x_1, x_2) := \langle 0|\,\mathsf{T}\,j^\Gamma_a(x)\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)|0\rangle\,, \tag{3.1.76}$$

with $j^\Gamma \in \{V^\mu, A^\mu, S, P\}$. This is the *full* three-point function, which is the same as the 1PI vertex with external quark propagators attached (i.e., to obtain the vertex, multiply with inverse quark propagators from the left and right).

■ The **quark-vector vertex** describes the coupling of the quark to a vector current. An example is the quark-photon vertex, which is the fundamental quantity that appears whenever a quark inside a hadron interacts with a photon.

■ The **quark-axialvector vertex** encodes its coupling to an axialvector current (e.g., the W- and Z-boson interactions with quarks are linear combinations of vector and axialvector vertices).

In the vector and axialvector cases, the two- and three-point functions above are related by a WTI which follows from Eq. (3.1.74):

$$\begin{aligned}\partial^x_\mu\, G^\mu(x, x_1, x_2) = {}& \langle 0|\,\mathsf{T}\,(\partial_\mu\, j^\mu(x))\;\psi(x_1)\,\overline{\psi}(x_2)\,|0\rangle \\ & + \delta(x^0 - x^0_1)\langle 0|\,\mathsf{T}\,\big[j^0(x), \psi(x_1)\big]\,\overline{\psi}(x_2)\,|0\rangle \\ & + \delta(x^0 - x^0_2)\langle 0|\,\mathsf{T}\psi(x_1)\,\big[j^0(x), \overline{\psi}(x_2)\big]\;|0\rangle\,.\end{aligned} \tag{3.1.77}$$

If we insert the respective commutator (3.1.69) for each type of current, we reproduce the quark propagator together with a flavor generator and a $\delta-$function. Employing vector current conservation $\partial_\mu V^\mu_a = 0$ and the PCAC relation $\partial_\mu A^\mu_a = 2mP_a$ (for equal quark masses), we obtain the **vector** and **axialvector WTIs:**

$$\partial^x_\mu\, G^\mu_V(x, x_1, x_2) = -\delta^4(x - x_1)\,\mathsf{t}_a\,S(x_1, x_2) + \delta^4(x - x_2)\,S(x_1, x_2)\,\mathsf{t}_a\,, \tag{3.1.78}$$

$$\begin{aligned}\partial^x_\mu\, G^\mu_A(x, x_1, x_2) = {}& 2m\,G_P(x, x_1, x_2) \\ & - \delta^4(x - x_1)\,\gamma_5\,\mathsf{t}_a\,S(x_1, x_2) - \delta^4(x - x_2)\,S(x_1, x_2)\,\mathsf{t}_a\,\gamma_5\,.\end{aligned} \tag{3.1.79}$$

The quark propagator is a diagonal matrix in flavor space but with different entries for different flavors, so it does not necessarily commute with all flavor generators t_a.

These relations become more transparent in momentum space, where the derivative with respect to x becomes a contraction with the momentum $q = p_1 - p_2$:

$$iq_\mu \, G_V^\mu(p_1, p_2) = S(p_1)\, \mathsf{t}_a - \mathsf{t}_a \, S(p_2)\,, \tag{3.1.80}$$

$$iq_\mu \, G_A^\mu(p_1, p_2) = 2m \, G_P(p_1, p_2) - S(p_1)\, \mathsf{t}_a \, \gamma_5 - \gamma_5 \, \mathsf{t}_a \, S(p_2)\,. \tag{3.1.81}$$

In other words, the effect of classical symmetries in the QFT is that they constrain the longitudinal (better: *non-transverse*) parts of $(n+1)$-point functions with respect to q^μ from the corresponding n-point functions. The vector WTI can be solved to obtain the most general form of the vertex that is compatible with vector current conservation, apart from further transverse terms with respect to the momentum q^μ (more below). The axialvector WTI relates the longitudinal part of the axialvector vertex with the pseudoscalar vertex and the quark propagator. Here we considered only the flavor-octet axial current A_a^μ; in the flavor-singlet channel we would have an additional term from the axial anomaly. Similar relations can be derived for higher n-point functions.

To work out the Fourier transforms, note that n-point correlation functions only depend on $n-1$ spacetime coordinates due to translation invariance: $G(x_1, \dots x_n) = G(x_1 - X, \dots x_n - X)$. For example, for a two-point function $S(x_1, x_2)$ we can define total and relative coordinates by

$$x_1 = X + \frac{z}{2}\,, \quad x_2 = X - \frac{z}{2} \qquad \Leftrightarrow \qquad X = \frac{x_1 + x_2}{2}\,, \quad z = x_1 - x_2\,. \tag{3.1.82}$$

From the behavior of the field operators under translations, Eqs. (2.2.10–2.2.11), we find

$$S(x_1, x_2) = \langle 0|\, \mathsf{T}\, \phi(X + \tfrac{z}{2})\, \phi(X - \tfrac{z}{2})\, |0\rangle = \langle 0|\, \mathsf{T}\, \phi(\tfrac{z}{2})\, \phi(-\tfrac{z}{2})\, |0\rangle = S(z)\,. \tag{3.1.83}$$

Then the Fourier transform becomes

$$\int d^4x_1 \int d^4x_2 \, e^{i(p_1 x_1 - p_2 x_2)}\, S(x_1, x_2) = \int d^4X \int d^4z \, e^{i(PX + pz)}\, S(z) = (2\pi)^4\, \delta^4(P)\, S(p)\,, \tag{3.1.84}$$

where $P = p_1 - p_2$, $p = (p_1 + p_2)/2$ and the $\delta-$function ensures $P = 0$, $p_1 = p_2 = p$ so that

$$S(z) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipz}\, S(p)\,, \qquad S(p) = \int d^4z \, e^{ipz}\, S(z)\,. \tag{3.1.85}$$

For a three-point function $G(x, x_1, x_2)$ we add the coordinate $x = X - y$ to (3.1.82). Translation invariance implies

$$G(x, x_1, x_2) = \langle 0|\, \mathsf{T}\, \phi(-y)\, \phi(\tfrac{z}{2})\, \phi(-\tfrac{z}{2})\, |0\rangle = G(y, z) \tag{3.1.86}$$

and the Fourier transform becomes

$$\begin{aligned} \int d^4x \int d^4x_1 \int d^4x_2 \, e^{i(p_1 x_1 - p_2 x_2 - qx)}\, G(x, x_1, x_2) &= \int d^4X \int d^4z \int d^4y \, e^{i(PX + pz + qy)}\, G(y, z) \\ &= (2\pi)^4\, \delta^4(P)\, G(p, q)\,, \end{aligned}$$

where $P = p_1 - p_2 - q$, the average momentum is $p = (p_1 + p_2)/2$, and the $\delta-$function ensures $q = p_1 - p_2$. In the same way we can work out

$$\int d^4x_1 \int d^4x_2 \int d^4x \, e^{i(p_1 x_1 - p_2 x_2 - qx)} \begin{bmatrix} G^\mu(x, x_1, x_2) \\ \partial_\mu^x \, G^\mu(x, x_1, x_2) \\ \delta^4(x - x_1)\, S(x_1, x_2) \\ \delta^4(x - x_2)\, S(x_1, x_2) \end{bmatrix} = (2\pi)^4\, \delta^4(P) \begin{bmatrix} G^\mu(p_1, p_2) \\ iq_\mu \, G^\mu(p_1, p_2) \\ S(p_2) \\ S(p_1) \end{bmatrix}$$

to arrive at Eq. (3.1.80). Note that we use the notation $G(p_1, p_2)$ and $G(p, q)$ interchangeably to keep things transparent, and we employ the same symbol for $G(y, z)$ in coordinate space, but this does not mean that G is the same function of the arguments.

Let us rewrite the vector WTI (3.1.80) for the **vector vertex** $\Gamma_V^\mu(p_1,p_2)$ defined by

$$G_V^\mu(p_1,p_2) = S(p_1)\,\Gamma_V^\mu(p_1,p_2)\,S(p_2)\,. \tag{3.1.87}$$

If we multiply with $S(p_1)^{-1}$ from the left and $S(p_2)^{-1}$ from the right and denote $p_1 = p+q/2$, $p_2 = p-q/2$, the WTI becomes

$$q_\mu\,\Gamma^\mu(p,q) = iS\big(p+\tfrac{q}{2}\big)^{-1} - iS\big(p-\tfrac{q}{2}\big)^{-1}\,, \tag{3.1.88}$$

where we dropped the flavor matrices for simplicity. Inserting the decomposition (2.3.11) for the quark propagator, we obtain

$$\begin{aligned} q_\mu\,\Gamma^\mu(p,q) &= A(p_1^2)\,\big(\not{p}_1 - M(p_1^2)\big) - A(p_2^2)\,\big(\not{p}_2 - M(p_2^2)\big) \\ &= \big(A(p_1^2)-A(p_2^2)\big)\,\not{p} + \frac{A(p_1^2)+A(p_2^2)}{2}\,\not{q} - \big(B(p_1^2)-B(p_2^2)\big) \\ &= \Sigma_A\,\not{q} + 2p\cdot q\,\big(\Delta_A\,\not{p} - \Delta_B\big)\,. \end{aligned} \tag{3.1.89}$$

Here we defined $B(p^2) = A(p^2)\,M(p^2)$ and the average and difference quotient

$$\Sigma_A = \frac{A(p_1^2)+A(p_2^2)}{2}\,, \qquad \Delta_A = \frac{A(p_1^2)-A(p_2^2)}{p_1^2-p_2^2} = \frac{A(p_1^2)-A(p_2^2)}{2p\cdot q}\,, \tag{3.1.90}$$

which are regular for $q^\mu \to 0$, and likewise for Δ_B. As a result, we can split off the momentum q^μ and read off the **Ball-Chiu vertex**

$$\Gamma^\mu_{\text{BC}}(p,q) = \Sigma_A\,\gamma^\mu + 2p^\mu\,\big(\Delta_A\,\not{p} - \Delta_B\big)\,. \tag{3.1.91}$$

For a tree-level vertex with the replacements $A(p^2)\to Z_\psi$ and $M(p^2)\to m_B$, this expression becomes $\Gamma^\mu_{\text{BC}} \to Z_\psi\,\gamma^\mu$ as expected.

Instead of the singlet and octet currents V^μ and V^μ_a, we could also consider linear combinations of them such as the **electromagnetic current**, which couples to the quarks through the quark charge matrix Q, e.g. for three flavors:

$$\mathsf{Q} = \begin{pmatrix} q_u & 0 & 0 \\ 0 & q_d & 0 \\ 0 & 0 & q_s \end{pmatrix} = \mathsf{t}_3 + \frac{\mathsf{t}_8}{\sqrt{3}} \quad \Rightarrow \quad V^\mu_{\text{em}}(x) = \bar{\psi}\,\gamma^\mu\,\mathsf{Q}\,\psi = V_3^\mu + \tfrac{1}{\sqrt{3}}\,V_8^\mu\,. \tag{3.1.92}$$

The corresponding vertex is the **quark-photon vertex**, which satisfies the same WTI (3.1.80) if we replace $\mathsf{t}_a \to \mathsf{Q}$ and thus has the same form as above. In other words, once we know the quark propagator, we already know a great deal about the quark-photon vertex from symmetries alone. The full vertex can be written as $\Gamma^\mu = \Gamma^\mu_{\text{BC}} + \Gamma^\mu_T$ with $q_\mu\,\Gamma^\mu_T = 0$, where the transverse part is not constrained and carries the dynamics such as the vector-meson poles (more in Sec. 3.1.3).

Here one can also see that it is actually not the *longitudinal* part that is constrained by the WTI, because if we had split the vertex into $\Gamma^\mu = q^\mu\,\Gamma_L + \Gamma^\mu_T$ with $q_\mu\,\Gamma^\mu_T = 0$, we would have obtained

$$\Gamma_L = \frac{1}{q^2}\Big[iS\big(p+\tfrac{q}{2}\big)^{-1} - iS\big(p-\tfrac{q}{2}\big)^{-1}\Big] = \frac{1}{q^2}\big[\Sigma_A\,\not{q} + 2p\cdot q\,\big(\Delta_A\,\not{p} - \Delta_B\big)\big]\,, \tag{3.1.93}$$

which is singular for $q^\mu \to 0$ (and violates the **Ward identity** $\Gamma^\mu(p,0) = i\,\partial S(p)^{-1}/\partial p_\mu$ which follows from Eq. (3.1.88)). One can systematically work out the WTI constraints for n-point functions by constructing 'minimal' tensor bases that are free of kinematic singularities and constraints.

WTIs from the path integral. As already mentioned around Eq. (2.2.113), WTIs can also be derived in the path integral formalism. This is relatively straightforward to do for the **Abelian local** $U(1)$ gauge invariance in QED. The partition function in QED has the same form as in QCD,

$$Z[J,\bar\eta\,,\eta\,] = \int \mathcal{D}[A,\psi,\bar\psi\,]\; e^{i\left(S[A,\psi,\bar\psi\,]+S_{\mathrm{GF}}[A\,]+S_{\mathrm{C}}[A,\psi,\bar\psi\,]\right)}, \tag{3.1.94}$$

except there are no ghosts because the Faddeev-Popov determinant is independent of the photon field A^μ and can be pulled out of the path integral. The resulting gauge-fixing and source terms read

$$S_{\mathrm{GF}} + S_{\mathrm{C}} = \int d^4x \left[\frac{1}{2\xi}\, A^\mu\partial_\mu\partial_\nu A^\nu - J_\mu A^\mu - \bar\psi\,\eta - \bar\eta\,\psi\right]. \tag{3.1.95}$$

If we keep the sources fixed, then a gauge transformation is just a relabeling of fields under the integral and leaves the generating functional invariant. Since the QED action is gauge invariant, and assuming that the integral measure remains invariant as well, this only affects the gauge-fixing and source terms:

$$\begin{aligned}
Z[J,\bar\eta,\eta] &= \int \mathcal{D}A'\,\mathcal{D}\psi'\,\mathcal{D}\bar\psi'\; e^{i\left(S[A',\psi',\bar\psi']+S_{\mathrm{GF}}[A']+S_{\mathrm{C}}[A',\psi',\bar\psi']\right)} \\
&= \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{i\left(S[A,\psi,\bar\psi]+S_{\mathrm{GF}}[A\,]+S_{\mathrm{C}}[A,\psi,\bar\psi]\right)}\; e^{i(\delta S_{\mathrm{GF}}+\delta S_{\mathrm{C}})} \\
&= Z[J,\bar\eta,\eta]\,\left\langle e^{i(\delta S_{\mathrm{GF}}+\delta S_{\mathrm{C}})}\right\rangle_J \qquad\Rightarrow\qquad \langle\,\delta S_{\mathrm{GF}}+\delta S_{\mathrm{C}}\,\rangle_J = 0\,.
\end{aligned} \tag{3.1.96}$$

Inserting the infinitesimal gauge transformations (2.1.37) in the Abelian case,

$$\delta\psi = i\varepsilon\,\psi\,, \qquad \delta\bar\psi = -i\bar\psi\,\varepsilon\,, \qquad \delta A_\mu = \frac{1}{g}\partial_\mu\varepsilon\,, \tag{3.1.97}$$

and taking partial integrations to factor out $\varepsilon(x)$, we obtain

$$\langle\,\delta S_{\mathrm{GF}}+\delta S_{\mathrm{C}}\,\rangle_J = \int d^4x\,\varepsilon(x)\left\langle\frac{1}{g}\,\partial_\mu\left(J^\mu - \frac{1}{\xi}\,\Box A^\mu\right) + i\left(\bar\psi\,\eta - \bar\eta\,\psi\right)\right\rangle_J = 0\,. \tag{3.1.98}$$

Since $\varepsilon(x)$ is arbitrary, the integrand must vanish too. At this point there is also no longer a need to distinguish the classical fields from the averaged fields in the notation (ϕ versus φ in Eqs. (2.2.44) and (2.2.55)), so we simply write $\langle A^\mu\rangle_J = A^\mu$, $\langle\bar\psi\rangle_J = \bar\psi$ and $\langle\psi\rangle_J = \psi$:

$$\partial_\mu\left(J^\mu - \frac{1}{\xi}\,\Box\,A^\mu\right) + ig\left(\bar\psi\,\eta - \bar\eta\,\psi\right) = 0\,. \tag{3.1.99}$$

With the effective action $\Gamma[A,\psi,\bar\psi\,] = W[J,\bar\eta,\eta\,] + S_{\mathrm{C}}$ we can use (2.2.44) to transform this relation into a generating WTI for connected n-point functions by writing

$$A^\mu = -\frac{\delta W}{\delta J_\mu}\,, \qquad \bar\psi = \frac{\delta W}{\delta\eta}\,, \qquad \psi = -\frac{\delta W}{\delta\bar\eta}\,. \tag{3.1.100}$$

Here we took into account the Grassmann nature of the sources η and $\bar{\eta}$, i.e., η has to be permuted to the derivative operator which gives a minus sign. If we take further derivatives with respect to $\bar{\eta}$ and η, we arrive at the WTI for the connected three-point function. Alternatively, we can use (2.2.45) and convert the equation into a generating WTI for 1PI n-point functions by writing

$$J^\mu = \frac{\delta\Gamma}{\delta A_\mu}\,, \quad \bar{\eta} = -\frac{\delta\Gamma}{\delta\psi}\,, \quad \eta = \frac{\delta\Gamma}{\delta\overline{\psi}}\,. \tag{3.1.101}$$

In this case Eq. (3.1.99) becomes

$$\partial_\mu \left(\frac{\delta\Gamma}{\delta A_\mu} - \frac{1}{\xi}\,\Box\, A^\mu \right) + ig \left(\overline{\psi}\,\frac{\delta\Gamma}{\delta\overline{\psi}} + \frac{\delta\Gamma}{\delta\psi}\,\psi \right) = 0\,. \tag{3.1.102}$$

Taking two further derivatives with respect to $\overline{\psi}$ and ψ and setting all fields to zero yields the WTI for the 1PI fermion-photon vertex:

$$\partial_\mu \left(\frac{1}{g}\,\frac{\delta^3\Gamma}{\delta\psi(x_1)\,\delta\overline{\psi}(x_2)\,\delta A_\mu(x)} \right) = \frac{i\,\delta^2\Gamma}{\delta\psi(x_1)\,\delta\overline{\psi}(x_2)}\left[\delta^4(x-x_1) - \delta^4(x-x_2)\right]. \tag{3.1.103}$$

This is identical to Eq. (3.1.88) and says that the divergence of the vertex equals the difference of the inverse quark propagators. We can also take a derivative of Eq. (3.1.102) with respect to A^μ, which yields the WTI for the inverse photon propagator:

$$\partial_\mu\,\frac{\delta^2\Gamma}{\delta A_\mu(x)\,\delta A_\nu(y)} = \frac{1}{\xi}\,\Box\,\partial^\nu\,\delta^4(x-y)\,. \tag{3.1.104}$$

In momentum space, this entails that the longitudinal part of the propagator remains undressed, which proves our statement below Eq. (2.3.14) in the Abelian theory:

$$q_\mu (D^{-1})^{\mu\nu}(q) = iq^2\,\frac{q^\nu}{\xi}\,. \tag{3.1.105}$$

In principle one can derive WTIs also for **non-Abelian local** gauge symmetries, but they become very cumbersome and it is more convenient to use BRST invariance to obtain relations of the form (2.2.114). The quickest way to generate the **Slavnov-Taylor identities** is to apply the BRST transformation directly to the correlation functions, which must also be BRST-invariant since already the QCD action including the gauge-fixing terms is BRST-invariant. As an example, we derive the QCD version of Eq. (3.1.105) by starting from the BRST transformations in Eq. (2.2.109),

$$\boldsymbol{\delta}\psi = ic\,\psi\,, \quad \boldsymbol{\delta}\overline{\psi} = -i\overline{\psi}\,c\,, \quad \boldsymbol{\delta}A^\mu_a = \frac{1}{g}D^\mu_{ab}\,c_b\,, \quad \boldsymbol{\delta}c_a = -\tfrac{1}{2}f_{abc}\,c_b\,c_c\,, \quad \boldsymbol{\delta}\bar{c}_a = -\frac{B_a}{g} = \frac{\partial_\mu A^\mu_a}{g\xi}\,,$$

where we inserted the equations of motion $f_a[A] + \xi B_a = 0$ for B_a. Now consider the quantity

$$\frac{\partial}{\partial x^\mu}\,\boldsymbol{\delta}\,\langle A^\mu_a(x)\,\bar{c}_b(y)\rangle_J = \frac{1}{g}\,\langle \partial_\mu D^\mu_{ac}\,c_c(x)\,\bar{c}_b(y)\rangle_J + \frac{1}{g\xi}\,\partial^x_\mu\,\partial^y_\nu\,\langle A^\mu_a(x)\,A^\nu_b(y)\rangle_J\,. \tag{3.1.106}$$

For vanishing sources the l.h.s. is zero. In momentum space, the second term on the right is the contraction of the gluon propagator with $q_\mu q_\nu$. For the first term we insert the DSE for the ghost propagator obtained from Eq. (2.2.40):

$$\left\langle \frac{\delta S}{\delta\bar{c}_a(x)}\,\bar{c}_b(y) \right\rangle = \langle \partial_\mu D^\mu_{ac}\,c_c(x)\,\bar{c}_b(y)\rangle = i\delta^4(x-y)\,\delta_{ab}\,. \tag{3.1.107}$$

Thus, in momentum space we arrive at $q_\mu q_\nu D^{\mu\nu}_{ab}(q) = -i\xi\,\delta_{ab}$, which states that the longitudinal part of the gluon propagator remains undressed also with interactions.

Now what if we are instead interested in **global flavor symmetries**? Let us first check QED with a global $U(1)$ symmetry instead of a local one. In that case $\delta A^\mu = 0$ since ε is a constant, and we can no longer eliminate the integral in Eq. (3.1.98) but get instead:

$$\langle\, \delta S_{\rm GF} + \delta S_{\rm C}\,\rangle_J = i\varepsilon \int d^4x\; \big\langle \overline{\psi}\,\eta - \bar{\eta}\,\psi \big\rangle_J = 0\,. \tag{3.1.108}$$

This equation is correct but not very useful: In the context of Eq. (3.1.88) it only tells us that the integrated equation vanishes – or in momentum space, that the difference of propagators on the right-hand side vanishes if their momenta are equal ($q^\mu = 0$).

We can cure the problem by tricking the path integral into believing that it deals with a *local* symmetry instead of a global one. Suppose we start from the free quark Lagrangian in Eq. (3.1.13) without the quark-gluon vertex, which it is not relevant for the discussion:

$$\mathcal{L} = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi\,, \qquad Z[\,\eta,\bar{\eta}\,] = \int \mathcal{D}[\,\psi,\overline{\psi}\,]\; e^{i\left(S[\,\psi,\overline{\psi}\,]+S_{\rm C}\right)} \tag{3.1.109}$$

with source terms $S_C = -\int d^4x\,(\overline{\psi}\,\eta + \bar{\eta}\,\psi)$. The action $S[\psi,\,\overline{\psi}\,]$ is invariant under the global $SU(N_f)_V \times U(1)_V$ symmetry. We consider $U(1)_V$ for simplicity, whose flavor-singlet current $V^\mu = \overline{\psi}\,\gamma^\mu\,\psi$ is conserved. The idea is to add more source terms to the action and define appropriate gauge transformations for the source fields, so that the *total* action including all sources becomes *locally* gauge invariant with respect to $U(1)_V$. This means we need a covariant derivative; from Eq. (3.1.109) we only need to add a term $\overline{\psi}\,B\!\!\!/\,\psi = V\cdot B$ to establish local $U(1)$ gauge invariance:

$$Z[\,B,\eta,\bar{\eta}\,] = \int \mathcal{D}[\,\psi,\overline{\psi}\,]\; e^{i\left(S[\,\psi,\overline{\psi}\,]+V\cdot B+S_{\rm C}\right)}. \tag{3.1.110}$$

Here, B plays the role of the gauge field but it is a **'background field'** since it does not appear in the path integral measure and thus does not change the content of the QFT. From Eq. (2.1.37) we have $\delta B_\mu = \partial_\mu\varepsilon$ because we are dealing with an Abelian gauge symmetry (we set the irrelevant new coupling to 1). As a result, the sum $S[\,\psi,\,\overline{\psi}\,]+V\cdot B$ is locally gauge invariant. Finally, we also make $S_{\rm C}$ gauge invariant in itself by imposing appropriate gauge transformations $\delta\eta = i\varepsilon\,\eta$ and $\delta\bar{\eta} = -i\varepsilon\,\bar{\eta}$ for the source fields.

Now start from $Z[\,B,\eta,\bar{\eta}\,]$ and perform a gauge transformation to primed quantities $\{\psi',\overline{\psi}',B',\eta',\bar{\eta}'\}$. The total action is gauge-invariant and the path integral measure as well, so that also the partition function is invariant: $Z[\,B,\eta,\bar{\eta}\,] = Z[\,B',\eta',\bar{\eta}'\,]$. Next, relabel the fields ψ and $\overline{\psi}$ back to unprimed quantities and work out the transformation of B, η and $\bar{\eta}$ only:

$$\begin{aligned}
&\int d^4x\,\big\langle V\cdot\delta B - \overline{\psi}\,\delta\eta - \delta\bar{\eta}\,\psi\big\rangle_J \\
&= \int d^4x\,\big[\langle V^\mu\rangle_J\,\partial_\mu\varepsilon - i\varepsilon\,\big(\langle\overline{\psi}\rangle_J\,\eta - \bar{\eta}\,\langle\psi\rangle_J\big)\big] \\
&= -\int d^4x\,\varepsilon(x)\,\big[\partial_\mu\langle V^\mu\rangle_J + i\,\big(\langle\overline{\psi}\rangle_J\,\eta - \bar{\eta}\,\langle\psi\rangle_J\big)\big] = 0\,.
\end{aligned} \tag{3.1.111}$$

Once again, because $\varepsilon(x)$ is arbitrary, we arrive at

$$\partial_\mu \langle V^\mu\rangle_J + i\left(\langle\bar\psi\rangle_J\,\eta - \bar\eta\,\langle\psi\rangle_J\right) = 0\,. \tag{3.1.112}$$

In order to arrive at Eq. (3.1.78) including connected Green functions, replace

$$\langle V^\mu\rangle_J = -\frac{\delta W}{\delta B_\mu}\,, \qquad \langle\bar\psi\rangle_J = \frac{\delta W}{\delta\eta}\,, \qquad \langle\psi\rangle_J = -\frac{\delta W}{\delta\bar\eta} \tag{3.1.113}$$

and perform a partial integration. Since $\varepsilon(x)$ is again arbitrary one can remove the integral, and the resulting master WTI becomes

$$\partial_\mu\,\frac{\delta W}{\delta B_\mu} = i\left(\frac{\delta W}{\delta\eta}\,\eta + \bar\eta\,\frac{\delta W}{\delta\bar\eta}\right). \tag{3.1.114}$$

It has the same form as in our first attempt (3.1.108) except that now we have a new correlation function $\delta W/\delta B$ that incorporates the current. The vector WTI (3.1.78) follows from applying two further derivatives with respect to η and $\bar\eta$ and setting the sources to zero.

Renormalization of currents. So far we have only dealt with bare currents that we derived from the bare Lagrangian (3.1.13). However, if we included renormalization constants for the vector and axialvector currents, the current-algebra relations (3.1.66) would fix both of them to $Z^2 = Z = 1$. Hence, these currents stay unrenormalized, which entails

$$\begin{aligned} V^\mu_{\mathrm{B}} &= (\bar\psi\,\gamma^\mu\,\psi)_{\mathrm{B}} = Z_2\,(\bar\psi\,\gamma^\mu\,\psi)_{\mathrm{R}} = V^\mu_{\mathrm{R}}\,, \\ A^\mu_{\mathrm{B}} &= (\bar\psi\,\gamma^\mu\gamma_5\,\psi)_{\mathrm{B}} = Z_2\,(\bar\psi\,\gamma^\mu\gamma_5\,\psi)_{\mathrm{R}} = A^\mu_{\mathrm{R}}\,. \end{aligned} \tag{3.1.115}$$

On the other hand, those relations do not give us closed equations for the scalar and pseudoscalar densities. In that case we can exploit the fact that their divergences are proportional to the quark masses, e.g., from the PCAC relation:

$$\partial_\mu A^\mu_{\mathrm{B}} = (2mP)_{\mathrm{B}} \stackrel{!}{=} (2mP)_{\mathrm{R}} = \partial_\mu A^\mu_{\mathrm{R}} \quad\Rightarrow\quad P_{\mathrm{B}} = \frac{1}{Z_m}\,P_{\mathrm{R}}\,, \tag{3.1.116}$$

and consequently

$$P_{\mathrm{B}} = (\bar\psi\gamma_5\psi)_{\mathrm{B}} = Z_2\,(\bar\psi\gamma_5\psi)_{\mathrm{R}} = \frac{1}{Z_m}\,P_{\mathrm{R}}\,. \tag{3.1.117}$$

The same result follows for the scalar density. In summary, the renormalized currents are (we drop the label 'R'):

$$\begin{aligned} V^\mu &= Z_2\,\bar\psi\,\gamma^\mu\psi\,, & P &= Z_2 Z_m\,\bar\psi\gamma_5\psi\,, \\ A^\mu &= Z_2\,\bar\psi\,\gamma^\mu\gamma_5\,\psi\,, & S &= Z_2 Z_m\,\bar\psi\psi\,. \end{aligned} \tag{3.1.118}$$

3.1.3 Extracting hadrons from QCD

We have not yet talked about how we can actually extract **hadron properties** from QCD. How would you calculate the mass of a hadron in a QFT? In quantum mechanics the answer is clear: define a potential V and solve the Schrödinger equation $H\psi = E\psi$ to obtain the energy spectrum of the system. Once you know the wave function ψ, you can calculate matrix elements for observables. But what becomes of the Schrödinger equation in QFT? Earlier we argued that the well-defined objects in a QFT are the correlation functions and that they encode the full content of the theory. Therefore, they should also carry any possible information on hadrons. But how can we extract that information?

We already mentioned in Sec. 2.2.1 that hadrons are contained in the state space of QCD: $|\pi\rangle$, $|N\rangle$, ... are one-particle states with a well-defined mass, momentum $\boldsymbol{p}$ and other quantum numbers that reflect the symmetries of QCD (angular momentum, parity, flavor, etc.). As a consequence, they enter in the completeness relation

$$\mathbb{1} = \sum_\lambda \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \, |\lambda\rangle\langle\lambda| \,, \tag{3.1.119}$$

where the Lorentz-invariant integral weight implements the condition that each hadron is on its mass shell ($p^2 = m_\lambda^2$ or $E_p^2 = \boldsymbol{p}^2 + m_\lambda^2$). One should keep in mind that the state space of QCD is *enormous*: it can contain (unphysical) colored states, colorless 'one-particle' bound states like mesons and baryons but also glueballs, multiquark and multi-hadron states – also the C^{14} nucleus should be somewhere buried in it.

Hadrons generate poles. In principle, the extraction of hadron properties from QCD is based on the **spectral representation** (2.2.7), which is also closely related to the experimental situation. When we derive it for a two-point function, it tells us that the onshell states in the completeness relation produce poles in the propagator at $p^2 = m_\lambda^2$, where m_λ is the physical mass of the state, and the multiparticle states produce cuts which start at $p^2 = 4m_\lambda^2$ and extend to infinity. Unfortunately this does not quite work out in QCD: when we insert the completeness relation into a quark or gluon two-point function, a colored quark cannot create a colorless hadron. In other words, quark and gluon propagators cannot produce physical hadron poles.

Fortunately, the spectral representation is not limited to two-point functions. In general one can show that for a given correlation function

$$G(x_1, \dots x_r) = \langle 0 | \mathsf{T} \, \phi(x_1) \dots \phi(x_n) \, \phi(y_1) \dots \phi(y_r) \, |0\rangle \tag{3.1.120}$$

each one-particle state $|\lambda(p)\rangle$ with onshell momentum $p^2 = m_\lambda^2$ produces a pole, where the correlation function factorizes at the pole:

$$G(x_1, \dots x_r) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ipz} \left[\frac{i \, \Psi\big(\{x_i\}, p\big) \, \Psi^\dagger\big(\{y_j\}, p\big)}{p^2 - m_\lambda^2 + i\epsilon} + \text{finite} \right]. \tag{3.1.121}$$

This is true as long as the residues $\Psi\big(\{x_i\}, p\big)$ at the poles are nonzero. These residues are the transition elements between the vacuum and the onshell hadron and they are called **Bethe-Salpeter wave functions (BSWFs)**.

The proof goes as follows. We start from a general $(n+r)$-point function

$$G(x_1,\dots y_r) = \langle 0|\,\mathsf{T}\,\phi(x_1)\dots\phi(x_n)\,\phi(y_1)\dots\phi(y_r)\,|0\rangle\,. \tag{3.1.122}$$

Because we want to insert the completeness relation, we are only interested in the time orderings where all $x_i^0 > y_j^0$. We can separate this contribution by writing

$$G(x_1,\dots y_r) = \Theta_{xy}\,\langle 0|\,\mathsf{T}\,\{\phi(x_1)\dots\phi(x_n)\}\,\mathsf{T}\,\{\phi(y_1)\dots\phi(y_r)\}\,|0\rangle + (\dots)\,, \tag{3.1.123}$$

where $\Theta_{xy} := \theta\big(\min(x_1^0,\dots x_n^0) - \max(y_1^0,\dots y_r^0)\big)$ and $(\dots)$ contains the remaining time orderings. Inserting the completeness relation, this becomes

$$G(x_1,\dots y_r) = \sum_\lambda \frac{1}{(2\pi)^3}\int\frac{d^3k}{2E_\lambda}\,\Theta_{xy}\,\langle 0|\,\mathsf{T}\,\phi(x_1)\dots\phi(x_n)\,|\lambda\rangle\,\langle\lambda|\,\mathsf{T}\,\phi(y_1)\dots\phi(y_r)\,|0\rangle + (\dots)\,, \tag{3.1.124}$$

where $|\lambda\rangle$ is an onshell state with momentum $\boldsymbol{k}$ and energy $E_\lambda = \sqrt{\boldsymbol{k}^2 + m_\lambda^2}$.

For the correlation functions of the theory, which are vacuum-to-vacuum transition matrix elements, translation invariance entails that they do not depend on the total coordinate (see Eq. (3.1.83)). For a vacuum-to-hadron amplitude, the behavior of the field operators and one-particle states under translations $U(1,a)$,

$$U(1,a)\,\psi_\alpha(x)\,U(1,a)^{-1} = \psi_\alpha(x+a)\,,\quad U(1,a)\,|\lambda(p)\rangle = e^{ip\cdot a}\,|\lambda(p)\rangle\,,\quad U(1,a)\,|0\rangle = |0\rangle \tag{3.1.125}$$

entails that the dependence on the total coordinate can only enter through a phase. That is, if we write $x_i = X + x_i'$ and $y_i = Y + y_i'$, we can factor out the dependence on X and Y:

$$\begin{aligned}\langle 0|\,\mathsf{T}\,\phi(x_1)\,\dots\phi(x_n)\,|\lambda\rangle &= \langle 0|\,\mathsf{T}\,U(1,X)\,\phi(x_1')\,U(1,X)^{-1}\dots U(1,X)\,\phi(x_n')\,U(1,X)^{-1}\,|\lambda\rangle \\ &= \langle 0|\,\mathsf{T}\,\phi(x_1')\,\dots\phi(x_n')\,|\lambda\rangle\,e^{-ikX} = \Psi\big(\{x_i\},\boldsymbol{k}\big)\,e^{-ikX}\,,\end{aligned} \tag{3.1.126}$$

where the Bethe-Salpeter wave function $\Psi\big(\{x_i\},\boldsymbol{k}\big)$ only depends on $n-1$ coordinates and no longer on X. (For example, if we set $X = x_1$ it only depends on $x_2'\dots x_n'$.) Likewise,

$$\langle\lambda|\,\mathsf{T}\,\phi(y_1)\dots\phi(y_r)\,|0\rangle = \langle\lambda|\,\mathsf{T}\,\phi(y_1')\,\dots\phi(y_r')\,|0\rangle\,e^{ikY} = \Psi^\dagger\big(\{y_j\},\boldsymbol{k}\big)\,e^{ikY}\,. \tag{3.1.127}$$

Denoting $z = X - Y$, we also have

$$\min(x_1^0,\dots x_n^0) - \max(y_1^0,\dots y_r^0) = X^0 - Y^0 + \min({x_1'}^0,\dots {x_n'}^0) - \max({y_1'}^0,\dots {y_r'}^0) =: z^0 + \Delta \tag{3.1.128}$$

and the full correlation function becomes

$$G(x_1,\dots y_r) = \sum_\lambda \frac{1}{(2\pi)^3}\int\frac{d^3k}{2E_\lambda}\,\theta(z^0+\Delta)\,e^{-ikz}\,\Psi\big(\{x_i\},\boldsymbol{k}\big)\,\Psi^\dagger\big(\{y_j\},\boldsymbol{k}\big) + (\dots)\,. \tag{3.1.129}$$

Now we use the following representation of the step function:

$$\theta(x) = \int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{i}{\omega+i\epsilon}\,e^{-i\omega x} \tag{3.1.130}$$

and take the Fourier transform with respect to z:

$$\begin{aligned}\int d^4z\,e^{ipz}\,G(x_1,\dots y_r) = \sum_\lambda\int\limits_{-\infty}^{\infty}d\omega\,\frac{i}{\omega+i\epsilon}\,e^{-i\omega\Delta}\int\frac{d^3k}{2E_\lambda}\,&\underbrace{\frac{1}{(2\pi)^4}\int d^4z\,e^{i(p-k)z}\,e^{-i\omega z^0}}_{\delta^3(\boldsymbol{p}-\boldsymbol{k})\,\delta(p^0-E_\lambda-\omega)} \\ \times\,\Psi\big(\{x_i\},\boldsymbol{k}\big)\,\Psi^\dagger\big(\{y_j\},\boldsymbol{k}\big) + (\dots)\,.&\end{aligned} \tag{3.1.131}$$

When integrating over d^3k, the $\delta-$function sets $\boldsymbol{k} = \boldsymbol{p}$ and thus $E_\lambda = \sqrt{\boldsymbol{p}^2 + m_\lambda^2}$, so we arrive at

$$\int d^4z\,e^{ipz}\,G(x_1,\dots y_r) = \sum_\lambda\frac{i}{p^0 - E_\lambda + i\epsilon}\,\frac{e^{-i(p^0-E_\lambda)\Delta}}{2E_\lambda}\,\Psi\big(\{x_i\},\boldsymbol{p}\big)\,\Psi^\dagger\big(\{y_j\},\boldsymbol{p}\big) + (\dots)\,. \tag{3.1.132}$$

Furthermore, we can write

$$\frac{i}{p_0 - E_\lambda + i\epsilon}\,\frac{e^{-i(p^0-E_\lambda)\Delta}}{2E_\lambda} = \frac{i(p_0+E_\lambda)}{p_0^2 - E_\lambda^2 + i\epsilon}\,\frac{e^{-i(p^0-E_\lambda)\Delta}}{2E_\lambda} \approx \frac{i}{p^2 - m_\lambda^2 + i\epsilon}\,, \tag{3.1.133}$$

where we approximated $p_0 \approx E_\lambda$ in the vicinity of the pole. Thus we arrive at the final result

$$\int d^4z\, e^{ipz}\, G(x_1, \dots y_r) \xrightarrow{p^2 \to m_\lambda^2} \frac{i\,\Psi\big(\{x_i\},p\big)\,\Psi^\dagger\big(\{y_j\},p\big)}{p^2 - m_\lambda^2 + i\epsilon}\,. \tag{3.1.134}$$

Note that n and r can be different, which means we can squeeze in the completeness relation at *any* position in a matrix element $\langle 0|\,\mathsf{T}\,\phi(x_1)\,\phi(x_2)\,\dots|0\rangle$. As long as the BSWFs on both sides are non-zero so that there is a non-vanishing overlap with the onshell state $|\lambda\rangle$, this will produce a pole at $p^2 = m_\lambda^2$ in the form of a Feynman propagator.

Applied to QCD, this means that even though hadrons are color singlets and cannot produce poles in elementary two-point functions, they still generate poles in *higher* $n-$point functions, e.g. the quark-antiquark **four-point function** in Fig. 3.3:

$$G_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) = \langle 0|\mathsf{T}\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)\,\psi_\gamma(x_3)\,\overline{\psi}_\delta(x_4)|0\rangle\,. \tag{3.1.135}$$

Inserting a complete set of states will produce meson poles because a composite operator $\psi\overline{\psi}$ can produce color singlet quantum numbers ($3 \otimes \overline{3} = 1 \oplus 8$). In fact, the four-point function encodes the *complete* meson spectrum that is compatible with the flavor quantum numbers of the quarks. The corresponding BSWF of a meson reads

$$\begin{aligned}\langle 0|\,\mathsf{T}\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)\,|\lambda,a\rangle &= \langle 0|\,\mathsf{T}\psi_\alpha\big(\tfrac{z}{2}\big)\,\overline{\psi}_\beta\big(-\tfrac{z}{2}\big)\,|\lambda,a\rangle\, e^{-ip\cdot x} \\ &= \Psi^a_{\alpha\beta}(z,p)\, e^{-ip\cdot x}\,,\end{aligned} \tag{3.1.136}$$

where we set the total coordinate as $x = (x_1+x_2)/2$ and the relative one by $z = x_1 - x_2$, so that $x_1 = x + \frac{z}{2}$ and $x_2 = x - \frac{z}{2}$. Since the flavor quantum numbers of mesons are related to the $SU(N_f)$ generators t_a, we attached a flavor index a. Likewise, we would find baryon poles in the analogous quark six-point function and we could write down the analogous BSWF with three quark fields.

The BSWFs are not truly 'wave functions' in the quantum-mechanical sense since they transform under finite-dimensional, non-unitary representations of the Lorentz group and thus one cannot directly extract probability information from them. Depending on the total angular momentum J of the onshell hadron $|\lambda\rangle$, after splitting off polarization vectors (for $J=1$ states), Dirac spinors (for $J=\frac{1}{2}$ states) etc., they can be expanded in tensor bases just like the correlation functions in Eq. (2.3.7):

$$\Psi^{\mu\nu\dots}_{\alpha\beta\dots}\big(\{q_i\},p\big) = \sum_{i=1}^{N} f_i(q_1^2,\, q_2^2,\, q_1\cdot p, \dots, p^2 = m_\lambda^2)\,\tau_i\big(\{q_i\},p\big)^{\mu\nu\dots}_{\alpha\beta\dots}\,. \tag{3.1.137}$$

For example, Lorentz covariance and parity invariance settle the general structure of the BSWF for a pseudoscalar meson in momentum space:

$$\Psi^a_{\alpha\beta}(q,p) = \left[\gamma_5\left(f_1 + f_2\,\not{p} + f_3\,\not{q} + f_4\,[\not{q},\not{p}]\right)\right]_{\alpha\beta}\,\mathsf{t}_a\,. \tag{3.1.138}$$

Here p is the total momentum of the meson and q the relative momentum between the quark and antiquark. The $f_i(q^2, q\cdot p, p^2 = m_\lambda^2)$ are the Lorentz-invariant dressing functions which depend on all invariant momentum variables, and they contain the information about the meson in question.

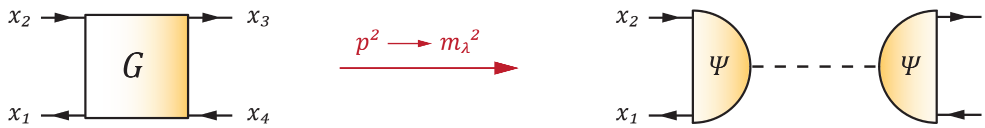

FIG. 3.3: Quark four-point function (3.1.135) and its separability at a particular meson pole according to Eq. (3.1.121). The dashed line is the Feynman propagator.

Current correlators. So how do we compute a hadron mass in practice? It appears that in order to extract the mass of a meson, we need to calculate the four-point function in Eq. (3.1.135) (or any other n-point function that creates meson poles) in some nonperturbative way and look for the poles in this quantity. While this is true in principle, it would also be a rather cumbersome endeavor: four-point functions are complicated objects, and moreover the ones above are not gauge-invariant since they contain quark fields with uncontracted color indices.

The advantage of Eq. (3.1.121) is that it is completely general and applies to *any* correlation function that has a non-vanishing overlap with the state $|\lambda\rangle$, in particular also those with *composite* operators. This is where the currents we defined in Eq. (3.1.23) become useful: Instead of working with the four-point function directly, we can simplify the problem by setting $x_1 = x_2 = x$ and $x_3 = x_4 = y$ and contracting the quark pairs with Dirac and flavor matrices $\mathsf{t}_a\,\Gamma_{\beta\alpha}$ and $\Gamma'_{\delta\gamma}\,\mathsf{t}_b$, where $\Gamma \in \{\gamma^\mu,\, \gamma^\mu\gamma_5,\, \mathbb{1},\, i\gamma_5,\, \dots\}$. In this way we obtain **current correlators**

$$\begin{aligned}
&(\mathsf{t}_a)_{ji}\,\Gamma_{\beta\alpha}\,\Gamma'_{\delta\gamma}\,(\mathsf{t}_b)_{lk}\,\langle 0|\mathsf{T}\,\psi_{\alpha i}(x)\,\overline{\psi}_{\beta j}(x)\,\psi_{\gamma k}(y)\,\overline{\psi}_{\delta l}(y)|0\rangle \\
&\qquad = \langle 0|\mathsf{T}\left\{\overline{\psi}(x)\,\Gamma\,\mathsf{t}_a\,\psi(x)\right\}\left\{\overline{\psi}(y)\,\Gamma'\,\mathsf{t}_b\,\psi(y)\right\}|0\rangle \\
&\qquad = \langle 0|\mathsf{T}\,j_a^\Gamma(x)\,j_b^{\Gamma'}(y)\,|0\rangle\,,
\end{aligned} \tag{3.1.139}$$

which are visualized in the upper panel in Fig. 3.4 and have the form

$$\langle 0|\mathsf{T}P_a(x)P_b(y)|0\rangle\,, \quad \langle 0|\mathsf{T}\,V_a^\mu(x)\,V_b^\nu(y)|0\rangle\,, \quad \langle 0|\mathsf{T}\,A_a^\mu(x)\,A_b^\nu(y)|0\rangle\,, \quad \text{etc.} \tag{3.1.140}$$

These are again two-point functions and can be viewed as effective meson propagators since they contain the composite fields P_a, V_a^μ, A_a^μ, etc. This is also a convenient way to filter the overwhelming information from the state space, because poles will only emerge from those states that coincide with the quantum numbers of the currents: a PP correlator produces pseudoscalar-meson poles, a VV correlator vector-meson poles and so on. Another advantage is that, in contrast to the four-point function with elementary quark field operators, the current correlators are also gauge-invariant since they contain gauge-invariant, local products of quark fields.

The pole residues of the current correlators are the BSWFs for $x_1 = x_2 = x$, i.e., $z = 0$ (which in momentum space means integration over the relative momentum), and contracted with the Dirac-flavor structures (i.e., taking Dirac and flavor traces):

$$\begin{aligned}
-\,(\mathsf{t}_a)_{ji}\,\Gamma_{\beta\alpha}\,\langle 0|\,\mathsf{T}\psi_{\alpha i}(x)\,\overline{\psi}_{\beta j}(x)\,|\lambda,b\rangle &= -\mathrm{Tr}\left\{\mathsf{t}_a\,\Gamma\,\Psi^b(0,p)\right\}\,e^{-ip\cdot x} \\
&= \langle 0|\,j_a^\Gamma(x)\,|\lambda,b\rangle = \langle 0|\,j_a^\Gamma(0)\,|\lambda,b\rangle\,e^{-ip\cdot x}\,.
\end{aligned} \tag{3.1.141}$$

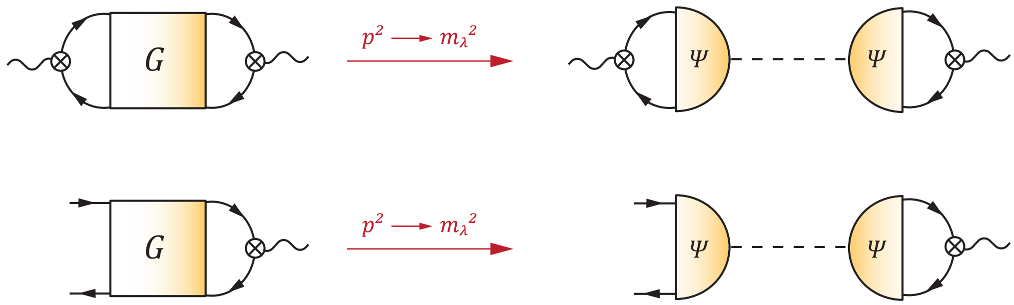

FIG. 3.4: Current correlators from Eq. (3.1.140) and three-point functions from (3.1.146). The symbol $\otimes$ represents the Dirac-flavor matrix $\Gamma\,\mathsf{t}_a$.

This is the vacuum-to-hadron transition element of the corresponding current. Take for example $\Gamma = \gamma^\mu\gamma_5$ and $i\gamma_5$, which produce axialvector and pseudoscalar currents, respectively. This restricts $|\lambda, a\rangle$ to pseudoscalar and axialvector mesons (for the moment we consider pseudoscalars only):

$$\langle 0|\, A_a^\mu(x)\, |\lambda, b\rangle = \delta_{ab}\, ip^\mu f_\lambda\, e^{-ip\cdot x}\,, \qquad \langle 0|\, P_a(x)\, |\lambda, b\rangle = \delta_{ab}\, r_\lambda\, e^{-ip\cdot x}\,. \tag{3.1.142}$$

The first quantity encodes the transition from a pseudoscalar meson to an axialvector current. By translation invariance the dependence on x goes into the phase, and the remainder is a Lorentz vector which can only depend on the onshell momentum p^μ with $p^2 = m_\lambda^2$, so the only possible tensor is p^μ. Since we also take the flavor trace of two generators, the only structure in flavor space is $\sim \delta_{ab}$, cf. (A.1.6). The remaining constant f_λ is the **electroweak decay constant** of the pseudoscalar meson: For example, the pion ($\lambda = \pi$) decays weakly into leptons ($\pi^+ \to W^+ \to \mu^+ + \nu_\mu$), so this defines the pion's electroweak decay constant f_π. The analogue r_λ for the pseudoscalar density is not directly measurable but will be useful in the following.

From here we can immediately derive a very useful relation. If we apply the PCAC relation (3.1.39) for equal quark masses, $\partial_\mu A_a^\mu(x) = 2mP_a(x)$, we obtain

$$f_\lambda\, m_\lambda^2 = 2m\, r_\lambda\,, \tag{3.1.143}$$

which is valid for all flavor non-singlet pseudoscalar mesons (in the singlet case there would be an additional term from the anomaly.) For example, it relates the pion decay constant and pion mass with the pseudoscalar transition matrix element r_π. If we go to the chiral limit and set $m = 0$, then the equation tells us that either the pion mass m_π or its decay constant f_π must vanish. This already resembles the Gell-Mann-Oakes-Renner (GMOR) relation, but so far we know nothing about spontaneous chiral symmetry breaking! The essence of the Goldstone theorem, which we will prove in Sec. 4.2, is that the pion decay constant f_π does *not* vanish in the chiral limit as a consequence of spontaneous chiral symmetry breaking, and thus the pion must be massless. Vice versa, the decay constants of excited pions ($m_\lambda \neq 0$) must vanish.

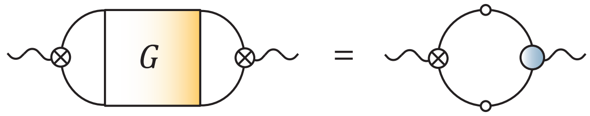

FIG. 3.5: Current correlator in terms of the quark propagator and corresponding vertex.

Since the current correlators are two-point functions, we can derive the spectral representation like in Eq. (2.2.15), e.g. for the pseudoscalar correlator ($z = x - y$),

$$\begin{aligned}
\langle 0|\mathsf{T}\, P_a(x)\, P_b(y)|0\rangle &= \Theta(z^0)\,\langle 0|\, P_a(x)\, P_b(y)\,|0\rangle + \Theta(-z^0)\,\langle 0|\, P_b(y)\, P_a(x)\,|0\rangle \\
&= \sum_\lambda \left[\int \frac{d^3p}{2E_{\boldsymbol{p}}}\,\frac{\Theta(z^0)\, e^{-ipz} + \Theta(-z^0)\, e^{ipz}}{(2\pi)^3}\right] r_\lambda^2\,\delta_{ab} \\
&= \sum_\lambda D_F(z, m_\lambda)\, r_\lambda^2\,\delta_{ab} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipz} \sum_\lambda \frac{i r_\lambda^2\,\delta_{ab}}{p^2 - m_\lambda^2 + i\varepsilon}\,,
\end{aligned} \tag{3.1.144}$$

or also mixed correlators:

$$\begin{aligned}
\langle 0|\mathsf{T}\, A_a^\mu(x)\, P_b(y)|0\rangle &= \sum_\lambda \left[\int \frac{d^3p}{2E_{\boldsymbol{p}}}\,\frac{\Theta(z^0)\, e^{-ipz} - \Theta(-z^0)\, e^{ipz}}{(2\pi)^3}\right] ip^\mu f_\lambda\, r_\lambda\,\delta_{ab} \\
&= -\frac{\partial}{\partial z_\mu} \sum_\lambda D_F(z, m_\lambda)\, f_\lambda\, r_\lambda\,\delta_{ab} = -\int \frac{d^4p}{(2\pi)^4}\, e^{-ipz} \sum_\lambda \frac{p^\mu f_\lambda\, r_\lambda\,\delta_{ab}}{p^2 - m_\lambda^2 + i\varepsilon}\,.
\end{aligned} \tag{3.1.145}$$

The sum over λ only goes over states which have an overlap with the pseudoscalar density, i.e., the pseudoscalar mesons, and in principle we should generalize the formulas to spectral densities which include the multiparticle contributions.

Since the result (3.1.121) is general, it also applies to three-point functions such as the ones in Eq. (3.1.76):

$$G^\Gamma_{a,\alpha\beta}(x, x_1, x_2) = \langle 0|\mathsf{T}\, j_a^\Gamma(x)\, \psi_\alpha(x_1)\, \overline{\psi}_\beta(x_2)|0\rangle\,. \tag{3.1.146}$$

This is just the four-point function contracted on one side only, as shown in Fig. 3.4. On the other hand, it is the vertex with quark propagators attached, e.g. in the vector case: $G_V^\mu(p_1, p_2) = S(p_1)\,\Gamma_V^\mu(p_1, p_2)\, S(p_2)$. This means that the vertex must also contain meson poles, but since its non-transverse part is fixed by the WTI these poles can only appear in the transverse part. Hence the quark-photon vertex must have transverse vector-meson poles, which is the origin of **'vector-meson dominance'**: when a photon couples to a quark, it fluctuates into ρ, ω, ... mesons.

With the same reasoning we can write the current correlator as in Fig. 3.5, since it is identical to the quark loop diagram with dressed quark propagators and the corresponding vertex. The VV correlator is also called **hadronic vacuum polarization** because it encodes the QCD contributions to the photon propagator. This quantity is experimentally accessible in the process $e^+e^- \to$ hadrons, and it is the biggest QCD contribution to the anomalous magnetic moment of the muon where the current Standard Model prediction deviates from experiment by $3\ldots 4\sigma$.

Finally, one can make repeated use of Eq. (3.1.121) also for higher n-point functions. An example is the quantity

$$\langle 0|\mathsf{T}\,\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2)\,j_c^\Gamma(x)\,\psi_\gamma(x_3)\,\overline{\psi}_\delta(x_4)\,|0\rangle\,, \tag{3.1.147}$$

where one can insert the completeness relation both to the left and the right of the current operator to produce BSWFs on either side. As a consequence, at the double pole location this becomes

$$\frac{i\,\Psi^a_{\alpha\beta}\big(\{q_i\},p\big)}{p^2-m_\lambda^2+i\varepsilon}\,\langle\lambda,a|\,j_c^\Gamma(0)\,|\lambda',b\rangle\,\frac{i\,\overline{\Psi}^{\dagger\,b}_{\gamma\delta}\big(\{q_i'\},p'\big)}{p'^2-m_{\lambda'}^2+i\varepsilon}\,. \tag{3.1.148}$$

The residue $\langle\lambda,a|\,j_c^\Gamma(0)\,|\lambda',b\rangle$ defines a hadron's **current matrix element**, such as for example the electromagnetic current matrix element $\langle\pi|V^\mu_{\text{em}}(0)|\pi\rangle$ which describes the coupling of the photon to a pion. The analogous n-point function for baryons contains $\langle N|V^\mu_{\text{em}}(0)|N\rangle$ which describes the electromagnetic coupling to the nucleon. The tensor decompositions of these matrix elements in analogy to Eq. (3.1.137) encode the various measurable **form factors** of hadrons: electromagnetic, axial, pseudoscalar, scalar form factors, etc., and we will return to them in Sec. 4.5.

Lattice QCD. Current correlators are frequently used in **lattice QCD** to compute the hadron spectrum. From the general formula (2.2.24) that relates a correlation function with the path integral, a current correlator can be calculated from

$$G(x-y)=\langle 0|\mathsf{T}\,j_1(x)\,j_2(y)\,|0\rangle=\frac{\int\mathcal{D}\phi\,e^{iS[\phi]}\,j_1(x)\,j_2(y)}{\int\mathcal{D}\phi\,e^{iS[\phi]}}\,. \tag{3.1.149}$$

We can write a generic Eucliden correlator in momentum space as

$$G_E(p_E)=\sum_\lambda\frac{R_\lambda}{p_E^2+m_\lambda^2}\,, \tag{3.1.150}$$

where $p_E^2=\boldsymbol{p}^2+p_4^2$, $E_\lambda^2=\boldsymbol{p}^2+m_\lambda^2$, and as usual the sum over λ is formal but suppose it contains one or a few isolated bound state poles at $p_E^2=-m_\lambda^2$. When we take a Fourier transform to $z_4>0$, then a pole in momentum space shows up as an exponential decay in Euclidean time:

$$G_E(z_4,\boldsymbol{p})=\sum_\lambda R_\lambda\int\frac{dp_4}{2\pi}\,\frac{e^{ip_4z_4}}{p_4^2+E_\lambda^2}=\sum_\lambda R_\lambda\,\frac{e^{-E_\lambda z_4}}{2E_\lambda}\,. \tag{3.1.151}$$

At large Euclidean times the mass m_0 of the ground state will dominate the sum, so one can extract m_0 from

$$-\lim_{z_4\to\infty}\frac{1}{z_4}\,\ln G_E(z_4,\boldsymbol{p}=0)=m_0\,. \tag{3.1.152}$$

The discretization of spacetime in lattice QCD and the restriction to a finite volume comes with a number of technical subtleties. For example, in a finite volume the multiparticle continuum turns into a series of discrete poles in p_E^2 (scattering states), which means that the energy levels computed on the lattice are not directly related to the masses of unstable hadrons above open thresholds. The formalism that allows one to relate the energy levels in a finite box to the pole positions of resonances in the complex plane is called the **Luescher method**.

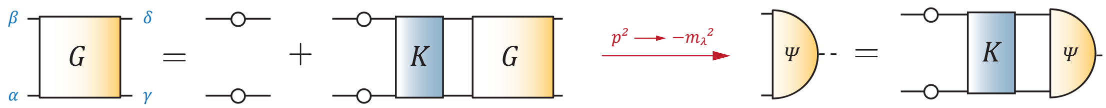

FIG. 3.6: Bethe-Salpeter equation for the four-point function and corresponding homogeneous equation for the Bethe-Salpeter wave function.

Bethe-Salpeter equations. Another way to extract hadron observables from QCD is to start from elementary correlation functions such as the four-point function (3.1.135) and write down a **Bethe-Salpeter equation (BSE)** for it. It has the schematic form shown in Fig. 3.6:

$$G = G_0 + G_0\, K\, G\,, \tag{3.1.153}$$

which is also called Dyson equation. Each multiplication stands for a four-dimensional integration in momentum space, so this is an integral equation for G, where G_0 is the disconnected part and K the kernel of the equation. The structure of the equation is similar to Eq. (2.2.65), where each step in

$$G = G_0 + G_0\, K\, G = G_0 + G_0\, K\, G_0 + G_0\, K\, G_0\, K\, G = \dots \tag{3.1.154}$$

is exact and gives $G^{-1} = G_0^{-1} - K$, whereas a perturbative series would only converge to that result if K is 'small' enough. The kernel can be modelled (e.g. by a ladder approximation which amounts to a gluon exchange between quark and antiquark) but in principle also systematically expanded in terms of the underlying correlation functions such as the quark and gluon propagators, three-point vertices, etc.

If G admits hadronic poles, then at the pole location it factorizes according to Eq. (3.1.121) and one arrives at the homogeneous BSE for the Bethe-Salpeter wave function:

$$\Psi = G_0\, K\, \Psi\,. \tag{3.1.155}$$

Thus, also here one does not actually need to calculate the four-point function directly in order to extract the pole locations. In practice the homogeneous BSE is an eigenvalue equation because it has the form $(G_0\, K)\, \Psi_\lambda = \eta_\lambda\, \Psi_\lambda$, where the η_λ are the eigenvalues of $G_0\, K$. The masses of ground and excited states can then be read off from the conditions $\eta_\lambda(p^2 = m_\lambda^2) = 1$. If the poles lie above thresholds, then they move into the complex plane onto higher Riemann sheets, but this does not invalidate the equation which still holds at the resonance pole location through analytic continuation, i.e., the above condition has solutions in the complex plane of p^2. Analogous BSEs can be derived for baryons (the three-body versions are also called **Faddeev equations**) and higher multiquark systems.