3.2 Hadron spectrum

We have studied the flavor structure of the QCD Lagrangian and its group-theoretical implications for hadron properties as well as for currents and for $n-$point functions. Now it is time for a reality check, because in principle the various symmetries of the Lagrangian should be reflected in the hadron spectrum:

▪ **SU(3) color gauge invariance:** Hadrons must be colorless, so they can only appear in the singlet representation of $SU(3)_c$. Color singlets can be obtained by combining quarks and antiquarks to **mesons** or three quarks to **baryons**:

$$\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\,,\qquad \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,. \tag{3.2.1}$$

Color singlets also arise from combining two (or more) gluons, which leads to the notion of **glueballs**:

$$\mathbf{8}\otimes\mathbf{8}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27}\,. \tag{3.2.2}$$

The product representations of $SU(N)$ are easiest to work out using Young diagrams (see Appendix A.3). Moreover, color singlets also appear in higher patterns of these combinations such as **tetraquarks**,

$$\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}=(\mathbf{1}\oplus\mathbf{8})\otimes(\mathbf{1}\oplus\mathbf{8})=\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27}\,, \tag{3.2.3}$$

pentaquarks ($\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}$), **hybrid mesons** ($\mathbf{3}\otimes\overline{\mathbf{3}}\otimes\mathbf{8}$), and so on. If we change the number of colors N_c, the nature of a 'hadron' will change as well (see Table A.2 in the appendix):

$$N_c\otimes\overline{N}_c=\mathbf{1}\oplus\dots\,,\qquad \underbrace{N_c\otimes\cdots\otimes N_c}_{N_c\text{ times}}=\mathbf{1}\oplus\dots\,, \tag{3.2.4}$$

which means that mesons still survive as $q\bar{q}$ states while baryons become bound states of N_c quarks instead of three.

▪ **Flavor symmetries:** The usual $SU(N_f)_V$ flavor symmetry allows us to classify hadrons in flavor multiplets, where in contrast to color all combinations are allowed. In the three-flavor case, mesons form flavor singlets and octets whereas baryons come in singlets, octets and decuplets (see Fig. 3.7):

$$\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\,,\qquad \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\,. \tag{3.2.5}$$

The states within these multiplets are labeled by the third isospin component I_3 and the hypercharge Y (or equivalently, the strangeness), which are conserved quantum numbers even if the $SU(3)$ flavor symmetry is broken. In fact, the observation that hadrons appear in $SU(3)$ octet, decuplet and singlet representations but *not* in the fundamental one was the starting point for the development of the quark model.

The $U(1)_V$ symmetry, on the other hand, corresponds to the **baryon number**. Since there is no quantum number that distinguishes mesons from glueballs, tetraquarks or hybrids, these are strictly speaking all mesons ($B=0$), whereas pentaquarks are technically baryons ($B=1$).

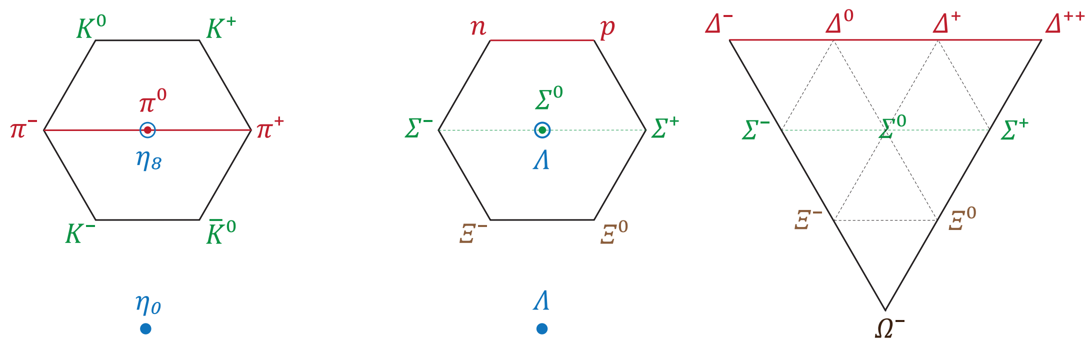


FIG. 3.7: $SU(3)_f$ meson singlet and octet (for 0^{-+} states); baryon singlet, octet and decuplet.

■ **Poincaré invariance:** The invariance of the QCD action under the Poincaré group gives us two quantum numbers to label the states, namely the eigenvalues of its Casimir operators: the **total angular momentum ('spin')** J and the **mass** M (see Appendix B). Together with parity invariance of the strong interaction, this allows us to arrange hadrons according to their J^P quantum numbers. We find scalar (0^+), pseudoscalar (0^-), vector (1^-), axialvector (1^+), tensor (2^+) mesons and more, whereas the possible J^P values for baryons are $\frac{1}{2}^\pm$, $\frac{3}{2}^\pm$, $\frac{5}{2}^\pm$, etc.

■ **Charge-conjugation invariance:** Charge conjugation exchanges a particle with its antiparticle and therefore reverses n_q for all flavors, the number of quarks minus antiquarks: $U_c\,|n_u, n_d, n_s, \dots\rangle = |-n_u, -n_d, -n_s, \dots\rangle$. Since B, I_3, Y and Q are then reversed as well, only states for which all these additive quantum numbers vanish can be $C-$parity eigenstates. These are the neutral flavorless mesons, which are their own antiparticles and can be classified according to J^{PC}. Applying U_c twice reverts the state back to its original one ($U_c^2 = 1$), so its possible eigenvalues are $C = \pm 1$. From the transformation properties of the quark fields,

$$U_c\,\psi_\alpha\,U_c^{-1} = \eta_c\,\overline{\psi}_\beta\,C_{\beta\alpha}\,, \qquad U_c\,\overline{\psi}_\alpha\,U_c^{-1} = \eta_c^\star\,C_{\alpha\beta}\,\psi_\beta \qquad C = i\gamma^2\gamma^0, \tag{3.2.6}$$

together with their anticommutativity, one can show that the Lagrangian is charge-conjugation invariant (η_c is a phase factor). One can also work out the transformation behavior of the currents:

$$S \to S, \qquad P \to P, \qquad V^\mu \to -V^\mu, \qquad A^\mu \to A^\mu\,. \tag{3.2.7}$$

Therefore, the mesons that are created by these currents carry the quantum numbers $J^{PC} = 0^{++}$, 0^{-+}, 1^{--} and 1^{++}, respectively.

Experimentally, hadrons do indeed come in $J^{P(C)}$ multiplets. For three flavors, in each $J^{P(C)}$ channel one finds $SU(3)_f$ octets and singlets for mesons as well as octets, decuplets and singlets for baryons (which can mix, see below). The corresponding states are distinguished by their quantum numbers I_3 and Y. In addition, the multiplets form ground states and **radial excitations**, which are distinguished by the remaining 'quantum number' M, i.e., their mass. In the following we discuss the current experimental status on the hadron spectrum.

3.2.1 Mesons

$SU(3)$ multiplets. Let us start with the meson spectrum obtained from three light quark flavors u, d and s. We first discuss the $SU(3)$ multiplets and resulting flavor wave functions. A vector ψ that transforms under the **fundamental representation** of $SU(3)$ satisfies $\psi' = U\psi$. In a given basis $|j\rangle$ with $\langle i|j\rangle = \delta_{ij}$, this implies

$$\psi = \sum_k \psi_k\,|k\rangle \quad \Rightarrow \quad \psi_i' = \sum_i U_{ij}\,\psi_j\,, \quad U|j\rangle = \sum_k U_{kj}|k\rangle\,, \tag{3.2.8}$$

where $\langle i|\,U|j\rangle = U_{ij}$ are the matrix elements in that basis, such that

$$\psi' = U\psi = \sum_j \psi_j\,U|j\rangle = \sum_{j,k} U_{kj}\,\psi_j\,|k\rangle = \sum_k \psi_k'\,|k\rangle\,. \tag{3.2.9}$$

The same relations hold for the generators, where for later convenience we attach a hat to distinguish the basis-independent operators from their matrix elements:

$$\hat{\mathsf{t}}_a\,|j\rangle = \sum_k (\mathsf{t}_a)_{kj}\,|k\rangle\,, \quad \langle i|\,\hat{\mathsf{t}}_a|j\rangle = (\mathsf{t}_a)_{ij}\,. \tag{3.2.10}$$

In the fundamental representation the generators are proportional to the Gell-Mann matrices. The two Cartan generators

$$\mathsf{t}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{Y} = \frac{2}{\sqrt{3}}\,\mathsf{t}_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.2.11}$$

correspond to the third isospin component I_3 and the hypercharge Y and label the states inside the multiplet. From Eq. (3.2.10) we can work out their eigenvalues:

$$\begin{aligned} \hat{\mathsf{t}}_3\,|u\rangle &= \tfrac{1}{2}|u\rangle\,, & \hat{\mathsf{Y}}\,|u\rangle &= \tfrac{1}{3}|u\rangle\,, \\ \hat{\mathsf{t}}_3\,|d\rangle &= -\tfrac{1}{2}|d\rangle\,, & \hat{\mathsf{Y}}\,|d\rangle &= \tfrac{1}{3}|d\rangle\,, \\ \hat{\mathsf{t}}_3\,|s\rangle &= 0\,, & \hat{\mathsf{Y}}\,|s\rangle &= -\tfrac{2}{3}|s\rangle\,, \end{aligned} \tag{3.2.12}$$

or we can read them off directly from the matrices t_3 and Y using a Cartesian basis for $|u\rangle$, $|d\rangle$ and $|s\rangle$. The eigenvalues (I_3, Y) define the **weight vectors**,

$$\left(\tfrac{1}{2}\,,\,\tfrac{1}{3}\right)\ldots u\,, \qquad \left(-\tfrac{1}{2}\,,\,\tfrac{1}{3}\right)\ldots d\,, \qquad \left(0\,,\,-\tfrac{2}{3}\right)\ldots s\,, \tag{3.2.13}$$

from where we can draw the triplet in the (I_3, Y) plane (left panel in Fig. 3.8). The remaining generators

$$\hat{\mathsf{t}}_\pm = \hat{\mathsf{t}}_1 \pm i\hat{\mathsf{t}}_2\,, \qquad \hat{\mathsf{u}}_\pm = \hat{\mathsf{t}}_6 \pm i\hat{\mathsf{t}}_7\,, \qquad \hat{\mathsf{v}}_\pm = \hat{\mathsf{t}}_4 \pm i\hat{\mathsf{t}}_5 \tag{3.2.14}$$

are ladder operators which lead away from the origin in the (I_3, Y) plane and connect these states with each other, cf. Fig. 3.8:

$$\begin{aligned} \hat{\mathsf{t}}_+\,|d\rangle &= |u\rangle\,, & \hat{\mathsf{u}}_+\,|s\rangle &= |d\rangle\,, & \hat{\mathsf{v}}_+\,|s\rangle &= |u\rangle\,, \\ \hat{\mathsf{t}}_-\,|u\rangle &= |d\rangle\,, & \hat{\mathsf{u}}_-\,|d\rangle &= |s\rangle\,, & \hat{\mathsf{v}}_-\,|u\rangle &= |s\rangle\,. \end{aligned} \tag{3.2.15}$$

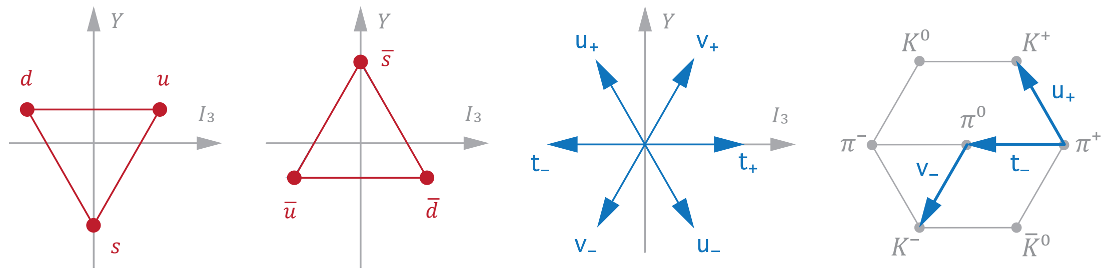


FIG. 3.8: Weight diagrams for the $SU(3)$ fundamental triplet and antitriplet in the (I_3, Y) plane and ladder operators. Right: Construction of the octet using ladder operators.

A vector in the **antitriplet representation $\overline{\mathbf{3}}$** transforms as

$$\psi'^{\dagger} = \psi^{\dagger}\, U^{\dagger} \quad \Leftrightarrow \quad \psi'^{\star}_i = U^{\star}_{ij}\, \psi^{\star}_j \,. \tag{3.2.16}$$

In this case the generators are given by $-\mathsf{t}_a^\star$, which satisfy the same commutation relations as the t_a,

$$[-\mathsf{t}_a^\star, -\mathsf{t}_b^\star\,] = i f_{abc}\,(-\mathsf{t}_a^\star)\,, \tag{3.2.17}$$

and thus define another three-dimensional representation (the conjugate representation of the group). For the group $SU(2)$, the generators t_a and $-\mathsf{t}_a^*$ are related by a unitary transformation and hence equivalent ($SU(2)$ representations are pseudoreal), but this is no longer true for $SU(N)$ with $N > 2$.

Writing the basis as $|\bar{j}\rangle$, we have

$$\hat{\mathsf{t}}_a\,|\bar{j}\rangle = \sum_k (-\mathsf{t}_a^*)_{kj}\,|\bar{k}\rangle\,, \quad \langle\bar{i}|\,\hat{\mathsf{t}}_a|\bar{j}\rangle = (-\mathsf{t}_a^*)_{ij} \tag{3.2.18}$$

which entails

$$\begin{aligned} \hat{\mathsf{t}}_3\,|\bar{u}\rangle &= -\tfrac{1}{2}|\bar{u}\rangle\,, & \hat{\mathsf{Y}}\,|\bar{u}\rangle &= -\tfrac{1}{3}|\bar{u}\rangle\,, \\ \hat{\mathsf{t}}_3\,|\bar{d}\rangle &= \tfrac{1}{2}|\bar{d}\rangle\,, & \hat{\mathsf{Y}}\,|\bar{d}\rangle &= -\tfrac{1}{3}|\bar{d}\rangle\,, \\ \hat{\mathsf{t}}_3\,|\bar{s}\rangle &= 0\,, & \hat{\mathsf{Y}}\,|\bar{s}\rangle &= \tfrac{2}{3}|\bar{s}\rangle\,. \end{aligned} \tag{3.2.19}$$

The weight vectors (I_3, Y) are

$$\left(-\tfrac{1}{2}\,,\,-\tfrac{1}{3}\right)\ldots\bar{u}\,, \qquad \left(\tfrac{1}{2}\,,\,-\tfrac{1}{3}\right)\ldots\bar{d}\,, \qquad \left(0\,,\,\tfrac{2}{3}\right)\ldots\bar{s} \tag{3.2.20}$$

and produce the inverted triangle in Fig. 3.8. The ladder operators work as before except the representation matrices of $\hat{\mathsf{t}}_\pm$ are $(-\mathsf{t}_1^*) \pm i(-\mathsf{t}_2^*) = -\mathsf{t}_\mp^*$ and not $-\mathsf{t}_\pm^*$, and similarly for the remaining ones, because due to the complex conjugation these are antilinear operators. As a result,

$$\begin{aligned} \hat{\mathsf{t}}_+\,|\bar{u}\rangle &= -|\bar{d}\rangle\,, & \hat{\mathsf{u}}_+\,|\bar{d}\rangle &= -|\bar{s}\rangle\,, & \hat{\mathsf{v}}_+\,|\bar{u}\rangle &= -|\bar{s}\rangle\,, \\ \hat{\mathsf{t}}_-\,|\bar{d}\rangle &= -|\bar{u}\rangle\,, & \hat{\mathsf{u}}_-\,|\bar{s}\rangle &= -|\bar{d}\rangle\,, & \hat{\mathsf{v}}_-\,|\bar{s}\rangle &= -|\bar{u}\rangle\,. \end{aligned} \tag{3.2.21}$$

Appendix A collects more information on the irreducible representations of $SU(N)$.

0^-	1^-	I	I_3	S		
π^+	ρ^+	1	1	0	$u\bar{d}$	t_+
π^0	ρ^0	1	0	0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	$\sqrt{2}\,\mathsf{t}_3$
π^-	ρ^-	1	-1	0	$d\bar{u}$	t_-
K^+	$K^{\star +}$	$1/2$	$1/2$	1	$u\bar{s}$	v_+
K^0	$K^{\star 0}$	$1/2$	$-1/2$	1	$d\bar{s}$	u_+
$\bar{K}^0$	$\bar{K}^{\star 0}$	$1/2$	$1/2$	-1	$s\bar{d}$	u_-
K^-	$K^{\star -}$	$1/2$	$-1/2$	-1	$s\bar{u}$	v_-
η_8	ω_8	0	0	0	$\frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$	$\sqrt{2}\,\mathsf{t}_8$
η_0	ω_0	0	0	0	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$	$\frac{1}{\sqrt{3}}\,\mathbb{1}$

TABLE 3.1: Normalized $SU(3)_f$ flavor wave functions for mesons.

Flavor wave functions for mesons. Next, we construct the irreducible **1** (singlet) and **8** (octet) representations along the lines of the discussion in App. A.3: We build the product wave functions as tensors of mixed rank $(1, 1)$ that transform under the reducible representation $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$, and by orthogonalizing them we single out the irreducible components in the end.

The simplest construction principle is the one via ladder operators illustrated in Fig. 3.8. If we start from $|\pi^+\rangle = |u\bar{d}\rangle$, then from Eqs. (3.2.12) and (3.2.19) the eigenvalues of $\hat{\mathsf{t}}_3$ and $\hat{\mathsf{Y}}$ are

$$\begin{aligned} \hat{\mathsf{t}}_3\,|u\bar{d}\rangle &= (\mathsf{t}_3 \otimes \mathbb{1} + \mathbb{1} \otimes (-\mathsf{t}_3^*))\;|u\bar{d}\rangle = |u\bar{d}\rangle\,, \\ \hat{\mathsf{Y}}\,|u\bar{d}\rangle &= (\mathsf{Y} \otimes \mathbb{1} + \mathbb{1} \otimes (-\mathsf{Y}^*))\;|u\bar{d}\rangle = 0\,, \end{aligned} \tag{3.2.22}$$

so the weight vector for $|u\bar{d}\rangle$ is $(I_3, Y) = (1, 0)$. If we apply the ladder operators $\hat{\mathsf{t}}_-$ and $\hat{\mathsf{u}}_+$, we obtain from Eqs. (3.2.15) and (3.2.21):

$$\hat{\mathsf{t}}_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle \sim |\pi^0\rangle\,, \qquad \hat{\mathsf{u}}_+|u\bar{d}\rangle = -|u\bar{s}\rangle \sim |K^+\rangle\,, \qquad \text{etc.} \tag{3.2.23}$$

These states are then normalized so that e.g. $\langle\pi^+|\pi^+\rangle = 1$. The remaining two states with $I_3 = 0$ and $Y = 0$ are constructed such that $|\eta_0\rangle$ is a singlet and $|\eta_8\rangle$ is orthogonal to $|\pi^0\rangle$ and $|\eta_0\rangle$. The resulting flavor wave functions are collected in Table 3.1.

Note that in a Cartesian basis the flavor wave functions are 3×3 matrices which are proportional to the $SU(N_f)$ generators: the π^+ wave function is $u \otimes \bar{d} = \mathsf{t}_+$, etc. Thus, the flavor wave functions for mesons can already be read off from the generators of the group, which goes back to Eq. (3.1.40): Since the currents and charges define representations of their algebra on the state space, the flavor content of the generators is inherited by the mesons that they create out of the vacuum. This is also the reason why we attached the group generators to the Bethe-Salpeter wave function (3.1.138), which should be read as the combinations that appear in Table 3.1.

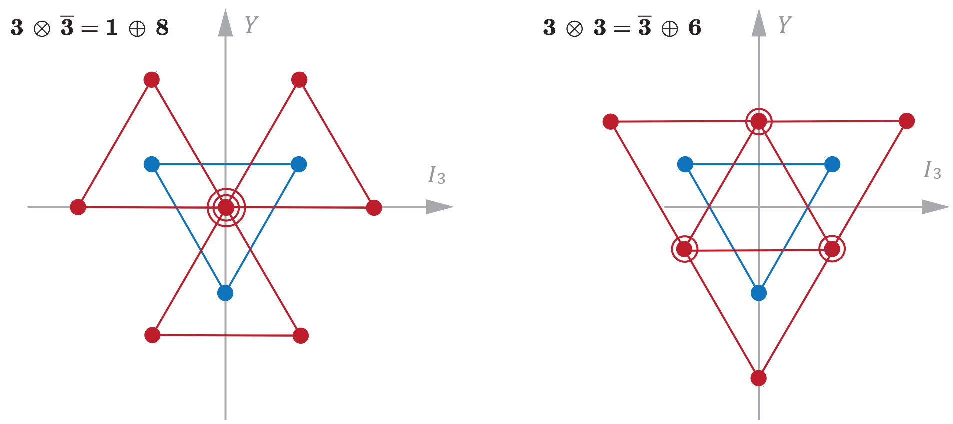


FIG. 3.9: Construction of weight diagrams by superimposing multiplets.

Yet another construction principle is shown in Fig. 3.9. Because the quantum numbers (I_3, Y) are additive, one can simply superimpose multiplets to arrive at the product states. For $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$, draw the triangle defined by $\mathbf{3}$ and add another $\overline{\mathbf{3}}$ at each corner of that triangle. This gives nine states for the corresponding values of (I_3, Y). Likewise, for $\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$, add another triplet $\mathbf{3}$ at each corner of $\mathbf{3}$ to arrive at the product states.

Mixing. Unfortunately, the identification of Table 3.1 with physical states only works out in the limit of exact $SU(3)$ flavor symmetry. If the quark masses $m_u = m_d = m_s$ are identical, the Lagrangian is invariant under $SU(3)_V$. As a consequence,

- All states in the multiplet have the same mass;
- All vector currents and charges are conserved;
- Not only the third isospin component I_3 and the hypercharge Y are conserved, but also the Casimirs of $SU(2)$ and $SU(3)$, which are the isospin I and the quantum numbers (p, q) that distinguish the multiplets (see Appendix A.2);
- The states π^0 (with $I = 1$) and η_8, η_0 (with $I = 0$), which have the same I_3 and Y, differ in at least one quantum number I or (p, q).

If $SU(3)_V$ is broken due to unequal quark masses, the states in the multiplets are no longer mass-degenerate and the $SU(3)$ Casimirs are no longer good quantum numbers. However, I_3 and Y are still conserved and commute with the Hamiltonian, so they can still be used to label the states. As a consequence, mesons carrying the same I_3 and S can mix with each other. This concerns for example the pseudoscalar mesons $\{\pi^0, \eta_8, \eta_0\}$ and the vector mesons $\{\rho^0, \omega_8, \omega_0\}$ which carry $I_3 = S = 0$: their flavor wave functions can mix with each other, and the mixed states are those that appear in the physical spectrum.

In principle the mixing effect can already be seen from the flavor matrix elements of the quark mass matrix M. Suppose we could write down an effective Hamiltonian of the form

$$H = H_0 + \mathsf{M}\,, \tag{3.2.24}$$

e.g. in the quark model or derived from some effective Lagrangian, where H_0 is flavor-independent and the quark mass operator is

$$\begin{aligned} \mathsf{M} &= (m_u\,|u\rangle\langle u| + m_d\,|d\rangle\langle d| + m_s\,|s\rangle\langle s|) \otimes \mathbb{1} \\ &+ \mathbb{1} \otimes \big(m_u\,|\bar{u}\rangle\langle\bar{u}| + m_d\,|\bar{d}\rangle\langle\bar{d}| + m_s\,|\bar{s}\rangle\langle\bar{s}|\big)\,. \end{aligned} \tag{3.2.25}$$

Applied to the flavor wave functions in Table 3.1, we find

$$\begin{aligned} \langle\pi^\pm\,|\,\mathsf{M}\,|\,\pi^\pm\rangle = \langle\pi^0\,|\,\mathsf{M}\,|\,\pi^0\rangle &= m_u + m_d\,, \\ \langle\eta_0\,|\,\mathsf{M}\,|\,\eta_0\rangle &= \frac{2}{3}\,(m_u + m_d + m_s)\,, \\ \langle\eta_8\,|\,\mathsf{M}\,|\,\eta_8\rangle &= \frac{1}{3}\,(m_u + m_d + 4m_s)\,, \end{aligned} \tag{3.2.26}$$

where the off-diagonal matrix elements are zero except for

$$\begin{aligned} \langle\eta_0\,|\,\mathsf{M}\,|\,\eta_8\rangle &= \frac{\sqrt{2}}{3}\,(m_u + m_d - 2m_s)\,, \\ \langle\pi_0\,|\,\mathsf{M}\,|\,\eta_8\rangle &= \frac{1}{\sqrt{2}}\,\langle\pi_0\,|\,\mathsf{M}\,|\,\eta_0\rangle = \frac{1}{\sqrt{3}}\,(m_u - m_d)\,. \end{aligned} \tag{3.2.27}$$

Because $m_u \approx m_d$, isospin symmetry is still approximately realized and the flavor breaking mostly comes from the strange-quark mass. Hence, the isospin I related to the Casimir of $SU(2)$ is approximately still a good quantum number, which leaves only a mixing for η_0 and η_8.

If we denote the flavor states generically by ψ_8 and ψ_0 and the mixed ones by ψ and ψ', we can define a mixing angle:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_0 \end{pmatrix} \xrightarrow{\text{ideal}} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \psi_8 \\ \psi_0 \end{pmatrix}. \tag{3.2.28}$$

In the case of **'ideal mixing'** we have $\cos\theta = 1/\sqrt{3}$, which leads to a separation into $SU(2)$ flavor wave functions, i.e., one state made of light quarks and another one made of strange quarks only:

$$\psi = \tfrac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d})\,, \qquad \psi' = s\bar{s}\,. \tag{3.2.29}$$

These diagonalize the mass matrix,

$$\langle\psi\,|\,\mathsf{M}\,|\,\psi\rangle = m_u + m_d\,, \qquad \langle\psi'\,|\,\mathsf{M}\,|\,\psi'\rangle = 2m_s\,, \qquad \langle\psi\,|\,\mathsf{M}\,|\,\psi'\rangle = 0\,, \tag{3.2.30}$$

and we find

$$\langle\psi\,|\,\mathsf{M}\,|\,\psi\rangle + \langle\psi'\,|\,\mathsf{M}\,|\,\psi'\rangle = \langle\psi_8\,|\,\mathsf{M}\,|\,\psi_8\rangle + \langle\psi_0\,|\,\mathsf{M}\,|\,\psi_0\rangle = m_u + m_d + 2m_s\,. \tag{3.2.31}$$

The actual mixing angles in the various meson channels are dynamical effects and have to be inferred from experiment (or computed by theory).

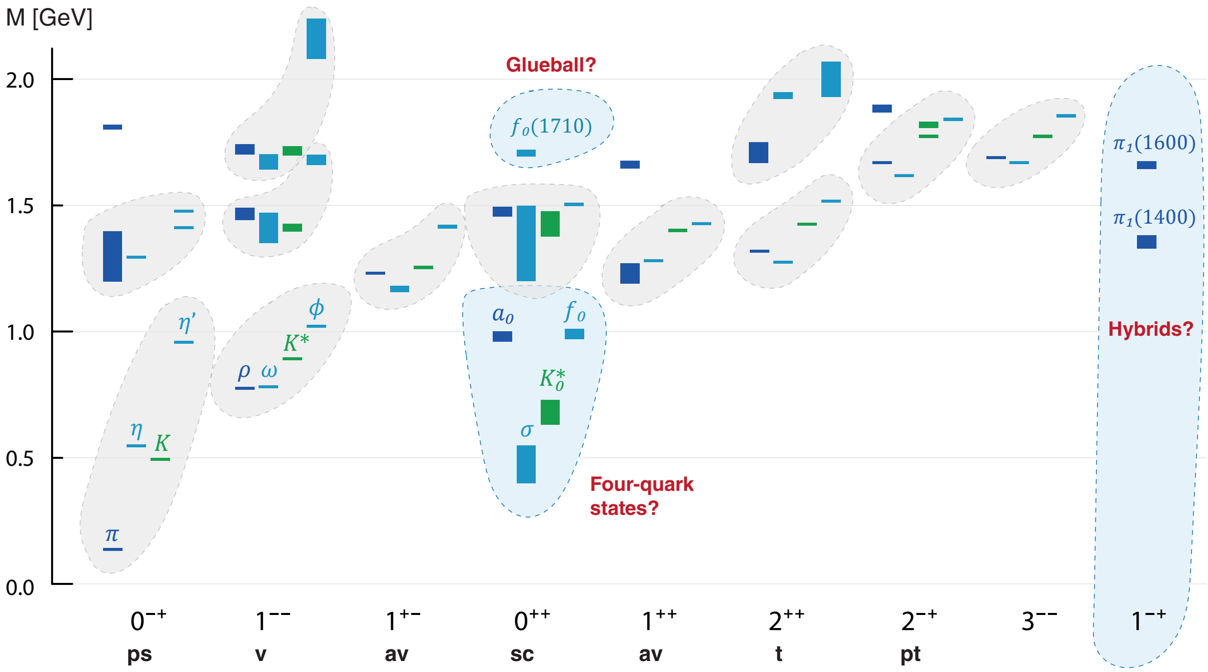


Fig. 3.10: Light and strange meson spectrum from the PDG (`https://pdglive.lbl.gov`).

Experimental spectrum. Now let us compare our expectations with the experimental spectrum. Fig. 3.10 shows the light meson spectrum from the PDG, where the bars are the quoted mass ranges. In each J^{PC} channel there are ground states and radial excitations. Each blob encloses a presumptive 'nonet' (i.e., octet plus singlet), where the states in light (dark) blue are those with $I = 0$ ($I = 1$) and the ones in green are the kaons. The naming scheme is as follows:

$$\begin{aligned} PC = -+: \quad & \{\pi,\, \eta,\, \eta'\}_J\,, \qquad & PC = ++: \quad & \{a,\, f,\, f'\}_J\,, \\ --: \quad & \{\rho,\, \omega,\, \phi\}_J\,, & +-: \quad & \{b,\, h,\, h'\}_J\,, \end{aligned} \tag{3.2.32}$$

where the subscript J is dropped for 0^{-+} and 1^{--} states. In addition, kaon-like states with $J^P = 0^+, 1^-, 2^+, 3^-, \dots$ are denoted by K^*.

A good channel to start with are the **vector mesons**, since this sets the prototype regarding expectations. Here we observe

$$m_\rho \approx m_\omega \qquad \text{and} \qquad m_\phi - m_{K^*} \approx m_{K^*} - m_\omega\,. \tag{3.2.33}$$

Suppose we have isospin symmetry ($m_u = m_d$) and ideal mixing, so that the ω is only made of u/d quarks and the ϕ is a pure $s\bar{s}$ state like in Eq. (3.2.29). If the dynamics were of the form (3.2.24), where the mass differences are entirely due to the different strange and u/d masses, then we would find:

$$\begin{aligned} m_\rho = m_\omega &= M_0 + 2m_u\,, \\ m_{K^*} &= M_0 + m_u + m_s\,, \\ m_\phi &= M_0 + 2m_s\,. \end{aligned} \tag{3.2.34}$$

M	I	S	0^{-+}	1^{--}	1^{+-}	0^{++}	1^{++}
8	1	0	$\pi\,(138)$ $\pi\,(1300)$ $\pi\,(1800)$	$\rho\,(770)$ $\rho\,(1450)$ $\rho\,(1700)$	$b_1\,(1235)$	$a_0\,(980)$ $a_0\,(1450)$	$a_1\,(1260)$ $a_1\,(1640)$
8, **1**	0	0	$\eta\,(548)$ $\eta'\,(958)$ $\eta\,(1295)$ $\eta\,(1405)$ $\eta\,(1475)$	$\omega\,(782)$ $\phi\,(1020)$ $\omega\,(1420)$ $\omega\,(1650)$ $\phi\,(1680)$ $\phi\,(2170)$	$h_1\,(1170)$ $h_1\,(1415)$	$f_0\,(500)$ $f_0\,(980)$ $f_0\,(1370)$ $f_0\,(1500)$ $f_0\,(1710)$	$f_1\,(1285)$ $f_1\,(1420)$
8	$\frac{1}{2}$	± 1	$K\,(495)$	$K^*\,(892)$ $K^*\,(1410)$ $K^*\,(1680)$	$K_1\,(1270)$	$K_0^*\,(700)$ $K_0^*\,(1430)$	$K_1\,(1400)$

M	I	S	2^{++}	2^{-+}	3^{--}	4^{++}	1^{-+}
8	1	0	$a_2\,(1320)$ $a_2\,(1700)$	$\pi_2\,(1670)$ $\pi_2\,(1880)$	$\rho_3\,(1690)$	$a_4\,(1970)$	$\pi_1\,(1400)$ $\pi_1\,(1600)$
8, **1**	0	0	$f_2\,(1270)$ $f_2'\,(1525)$ $f_2\,(1950)$ $f_2\,(2010)$ $f_2\,(2300)$ $f_2\,(2340)$	$\eta_2\,(1645)$ $\eta_2\,(1870)$	$\omega_3\,(1670)$ $\phi_3\,(1850)$	$f_4\,(2050)$	
8	$\frac{1}{2}$	± 1	$K_2^*\,(1430)$	$K_2\,(1770)$ $K_2\,(1820)$	$K_3^*\,(1780)$	$K_4^*\,(2045)$	

TABLE 3.2: Well-established light and strange mesons in terms of J^{PC}, isospin I and strangeness S (PDG 2020, https://pdglive.lbl.gov). Mesons with $I = S = 0$ belonging to different multiplets (M = **1** or **8**) can mix with each other, and in principle also the neutral members of the $I = 1$ states, so in these cases an identification with flavor-octet or singlet states is not possible. Note also that C parity is only a good quantum number for neutral mesons.

M_0 is some flavor-independent mass that depends on J^{PC} and the radial quantum number (otherwise they would be the same for each multiplet). Then Eq. (3.2.33) with the values from Table 3.2 yields $m_s - m_u \approx 120$ MeV, and we have the relation

$$m_\omega + m_\phi = 2m_{K^*} \tag{3.2.35}$$

which is realized to good extent in nature. Such empirical mass formulas are called **Gell-Mann-Okubo relations**.

In any case, for vector mesons ideal mixing seems to be well realized since the masses of $\{\rho, \omega\} \to K^* \to \phi$ differ roughly by one unit of the strange-quark mass. As one can see in Fig. 3.10, the pattern is (to a lesser extent) still visible in the 1^{+-}, 1^{++}, 2^{++} and some other channels, but there are two channels where the mass ordering does not work at all: the pseudoscalars 0^{-+} and the scalars 0^{++}. Apparently there are further mechanisms at play to which we turn now.

No parity doublets. In the chiral limit $m_u = m_d = m_s = 0$ the Lagrangian is invariant under a $SU(3)_V \times SU(3)_A$ chiral symmetry. In that case all mesons within a given J^{PC} multiplet would become mass-degenerate, and we expect **parity doublets** for mesons with same J but different P. Suppose $|\lambda\rangle$ is an eigenstate of the Hamiltonian with positive parity, so that

$$H\,|\lambda\rangle = E\,|\lambda\rangle\,, \qquad U_P\,|\lambda\rangle = +|\lambda\rangle\,. \tag{3.2.36}$$

Because the axial charge switches sign under parity,

$$\begin{aligned} U_P\,Q_A\,U_P^{-1} &= \int d^3x\,U_P\,\psi_\alpha^\dagger(x)\,U_P^{-1}\,(\gamma_5)_{\alpha\beta}\,U_P\,\psi_\beta(x)\,U_P^{-1} \\ &= \int d^3x\,\psi^\dagger(t,-\boldsymbol{x})\,\gamma^0\,\gamma_5\,\gamma^0\,\psi(t,-\boldsymbol{x}) \\ &= -\int d^3x\,\psi^\dagger(x)\,\gamma_5\,\psi(x) = -Q_A\,, \end{aligned} \tag{3.2.37}$$

the state $|\lambda'\rangle = Q_A\,|\lambda\rangle$ carries negative parity:

$$U_P\,|\lambda'\rangle = U_P\,Q_A\,|\lambda\rangle = -Q_A\,U_P\,|\lambda\rangle = -|\lambda'\rangle\,. \tag{3.2.38}$$

If chiral symmetry holds, the axial charge commutes with the Hamiltonian, $[Q_A, H] = 0$, and therefore $|\lambda'\rangle$ is an eigenstate with the same mass:

$$H\,|\lambda'\rangle = H\,Q_A\,|\lambda\rangle = Q_A\,H\,|\lambda\rangle = E\,|\lambda'\rangle\,. \tag{3.2.39}$$

Thus, chiral symmetry entails that the masses of the pseudoscalars (0^{-+}) are degenerate with the scalars (0^{++}), vector mesons (1^{--}) with axialvectors (1^{+-}), and so on.

Now, the three-flavor chiral symmetry is explicitly broken because $m_s \gg m_u \approx m_d$, but the two-flavor $SU(2)_V \times SU(2)_A$ symmetry should still approximately work since u and d quarks are almost massless. Hence we should still see remnants of this pattern in the spectrum. We do not, though: the pion is almost massless in contrast to its scalar partner and the vector mesons are much lighter than the axialvector ones.

On the other hand, the fact that $SU(2)_V$ still works out well (mesons with same isospin I have about the same mass) tells us that something is wrong with the $SU(N_f)_A$ part. Combined with the unnaturally light pseudoscalar mesons, these are the typical symptoms of a **spontaneous symmetry breaking** of $SU(N_f)_A$, which would produce $N_f^2 - 1$ massless Goldstone bosons in the chiral limit. The three pions are indeed almost massless ($m_\pi \approx 140$ MeV); the kaons are heavier but they also contain one strange quark, so they should feel the impact of explicit chiral symmetry breaking more strongly than the pions.

In Sec. 4.2 we will derive the Gell-Mann-Oakes-Renner relation, which states that the *squared* pseudoscalar meson masses are proportional to the quark masses. Based on this, we would interpret matrix elements such as $\langle\pi|\,\mathsf{M}\,|\pi\rangle = m_u + m_d$ to be proportional to m_π^2 instead of m_π (this becomes explicit in chiral perturbation theory). With Eq. (3.2.31), the analogue of the Gell-Mann-Okubo relation (3.2.35) then becomes

$$m_\eta^2 + m_{\eta'}^2 = m_{\eta_0}^2 + m_{\eta_8}^2 = 2m_K^2\,. \tag{3.2.40}$$

$\eta - \eta'$ mixing. If chiral symmetry is indeed spontaneously broken, it should break all axial symmetries, $SU(3)_A$ and $U(1)_A$. Hence we would expect nine Goldstone bosons, including the pions, the kaons and both the η_8 and η_0. Suppose we had ideal mixing: then η and η' would be the analogues of ω and ϕ in the vector channel,

$$\eta = \tfrac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d})\,, \qquad\qquad \eta' = s\bar{s}\,, \tag{3.2.41}$$

where the η is mass-degenerate with the pion and the η' (as a pure $s\bar{s}$ state) would acquire mass due to the explicit chiral symmetry breaking, similarly to the kaons. Assuming that Eq. (3.2.40) holds, then with $m_\eta = m_\pi$ we expect to find $m_{\eta'} \sim 690$ MeV. Going away from ideal mixing, the masses of η and η' should move in opposite directions: if $m_\eta > m_\pi$, we should find $m_{\eta'} \lesssim 690$ MeV. However, this is not realized at all: the η is heavier than the kaon ($m_\eta \approx 550$ MeV) and the η' mass is almost twice as large ($m_{\eta'} \approx 960$ MeV).

This argument and other ones lead us to believe that the $U(1)_A$ symmetry may not have been realized to begin with. It is still satisfied by the QCD Lagrangian in the chiral limit, but the classical symmetry is *anomalously* broken at the quantum level because it does not survive the quantization — this is the $U(1)_A$ **anomaly**. We already anticipated in Eq. (3.1.54) that the divergence of the axial current picks up an extra term. If we work out Eq. (3.1.143) in the isosinglet case, we obtain

$$f_{\eta_0} m_{\eta_0}^2 = 2\,\frac{m_u + m_d + m_s}{3}\,r_{\eta_0} + \frac{g^2 N_f}{(4\pi)^2}\,\langle 0|\,\widetilde{F}_a^{\mu\nu}(0)\,F_{\mu\nu}^a(0)\,|\eta_0\rangle\,. \tag{3.2.42}$$

Even if we set all quark masses to zero, the right-hand side of this equation remains nonzero and therefore also the η_0 remains massive in the chiral limit. Through a mixing with η_8, the extra term contributes to both η and η' masses. Another manifestation of this is the **Witten-Veneziano relation**, where χ_T is the so-called topological susceptibility:

$$m_\eta^2 + m_{\eta'}^2 = 2m_K^2 + \frac{4N_f\,\chi_T}{f_\pi^2}\,. \tag{3.2.43}$$

Missing exotics. Another observation in Fig. (3.10) is that not all J^{PC} quantum numbers appear: The **'exotic' quantum numbers** 0^{--}, 0^{+-}, 1^{-+} and 2^{+-} are absent from the light-meson spectrum, with the exception of the higher-lying $\pi_1(1400)$ and $\pi_1(1600)$ in the 1^{-+} channel.

The absence of exotic mesons can be understood from the nonrelativistic quark model. So far we have labeled $q\bar{q}$ states according to their J^{PC} eigenvalues. Now assume that the total spin S of the $q\bar{q}$ pair ($S = 0$ or $S = 1$) and its intrinsic orbital angular momentum $L = 0, 1, 2, \dots$ are also good quantum numbers. Then from the angular-momentum addition rules we have $|L - S| \leq J \leq L + S$, and we can motivate the following two relations:

$$P = (-1)^{L+1} \qquad \text{and} \qquad C = (-1)^{L+S} . \tag{3.2.44}$$

The first arises from the observation that a $q\bar{q}$ pair has intrinsic parity -1 and its spatial wave function has parity $(-1)^L$; parity is multiplicative, hence the factor $(-1)^{L+1}$. Charge conjugation exchanges quark and antiquark, so the value of C can be deduced by exchanging $q \leftrightarrow \bar{q}$ and then interchanging their positions and spins. The symmetry of the spin states is $(-1)^{S+1}$ because $S = 0$ is antisymmetric ($|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$) and $S = 1$ symmetric ($|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$). The factor $(-1)^L$ is as before, and a minus sign comes from interchanging the fermions. The combined operation gives $C = -(-1)^L (-1)^{S+1} = (-1)^{L+S}$.

The first relation above says that states with alternating L have alternating parity, and the second one entails that once L and S are specified, C (and thus J^{PC}) is fixed as well. These rules are quite efficient for cataloguing the possible J^{PC} combinations:

- $L = 0$ are orbital ground states (s waves) and should therefore correspond to the lightest mesons. According to (3.2.44) they must have negative parity. $S = 0$ gives us the pseudoscalars 0^{-+}; from $S = 1$ we obtain the vector mesons 1^{--}.
- $L = 1$ are orbital excitations (p waves) with positive parity. From $S = 0$ we obtain the axialvectors 1^{+-} and from $S = 1$ we get scalar (0^{++}), axialvector (1^{++}) and tensor mesons (2^{++}).
- From $L = 2$ (d waves) we obtain further vectors (1^{--}) plus states with $J = 2$ and $J = 3$, and so on for higher L.

The resulting mass ordering is shown in Fig. 3.11. Pseudoscalars and vectors are the lightest mesons because they are in an orbital s wave. Since they carry different quark spin S, their mass splitting is generated by spin-spin interactions between quark and antiquark. This is called **'hyperfine splitting'** because of its analogy to the hydrogen atom, where the hyperfine structure is caused by the coupling between electron and proton spin. All other mesons are orbitally excited because they carry higher L. For the lowest-lying states we should thus expect a mass pattern

$$\{0^{-+}\} \;<\; \{1^{--}\} \;<\; \{1^{+-}\} \;\lesssim\; \{0^{++},\, 1^{++},\, 2^{++}\} \;\lesssim\; \dots \tag{3.2.45}$$

which is also how we arranged the columns in Fig. 3.10. Also frequently used is the **spectroscopic notation** ${}^{2S+1}L_J$ with $L = S, P, D, F, \dots$, where this becomes

$$\{{}^1S_0\} \;<\; \{{}^3S_1\} \;<\; \{{}^1P_1\} \;\lesssim\; \{{}^3P_0,\, {}^3P_1,\, {}^3P_2\} \;\lesssim\; \dots \tag{3.2.46}$$

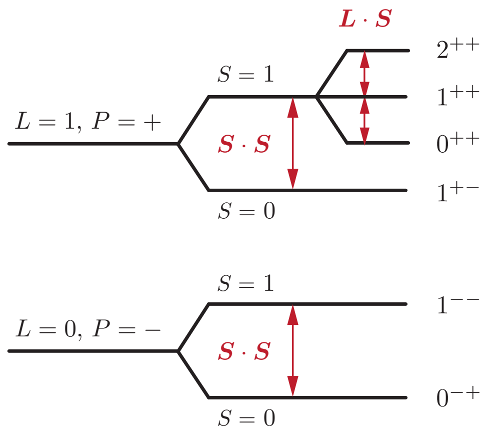


Fig. 3.11: Expected J^{PC} level ordering for mesons.

Interestingly, the analysis forbids exotic quantum numbers 0^{--}, 0^{+-}, 1^{-+} and 2^{+-}. If such states were observed, we would then conclude that they are not made of $q\bar{q}$ but something else. The only known examples in the light-meson spectrum are the $\pi_1(1400)$ and $\pi_1(1600)$ with 1^{-+}. They are candidates for **hybrid mesons**, i.e., states with valence quarks and gluons, because the combination $q\bar{q}g$ produces the quantum numbers 1^{-+} naturally (among others). From the general formula (3.1.121) we could look for hybrid mesons in higher n-point functions with gluonic content or, equivalently, current correlators of the form

$$\langle 0|\mathsf{T} j(x) j(y)|0\rangle \qquad \text{e.g. with} \qquad j(x) = \bar{\psi}(x)\, \not{D}\, \psi(x)\,, \tag{3.2.47}$$

where $\bar{\psi}\not{D}\,\psi$ is the simplest gauge-invariant combination that involves gluon fields. Such calculations are being done in lattice QCD and they find indeed additional states which do not show up when using $\bar{\psi}\psi$ operators only.[2]

On the other hand, Eq. (3.1.121) states that a meson pole will appear in *any* correlation function that has non-zero overlap with the state. It turns out that Bethe-Salpeter wave functions of the form (3.1.136) do not vanish for exotic quantum numbers even though they only contain quark and antiquark operators. As a consequence, poles with exotic quantum numbers can also show up in the quark-antiquark four-point function in Fig. 3.3. This goes back to the observation that the relations (3.2.44) are nonrelativistic, because P and C are conserved quantum numbers whereas L and S are not. Only J corresponds to a Casimir operator of the Poincaré group; S and L are not Poincaré-invariant and can mix in different reference frames. This is also why only J^{PC} should be used to label multiplets. For example, from the nonrelativistic analysis above the pion should carry $L=0$, but the pion's relativistic BSWF in Eq. (3.1.138) also contains $L=1$ components, namely the tensors $\not{q}$ and $[\not{q},\not{P}]$ which depend on the relative momentum q and thus correspond to p waves in the pion's rest frame. (This is analogous to the 'lower components' in Dirac spinors which come about through relativity.) Similary, at the level of BSWFs, exotic mesons are not generally forbidden as $q\bar{q}$ states but merely do not survive the non-relativistic limit.

[2] J. J. Dudek, Phys.Rev.D 84 (2011) 074023, `arXiv:1106.5515 [hep-ph]`.

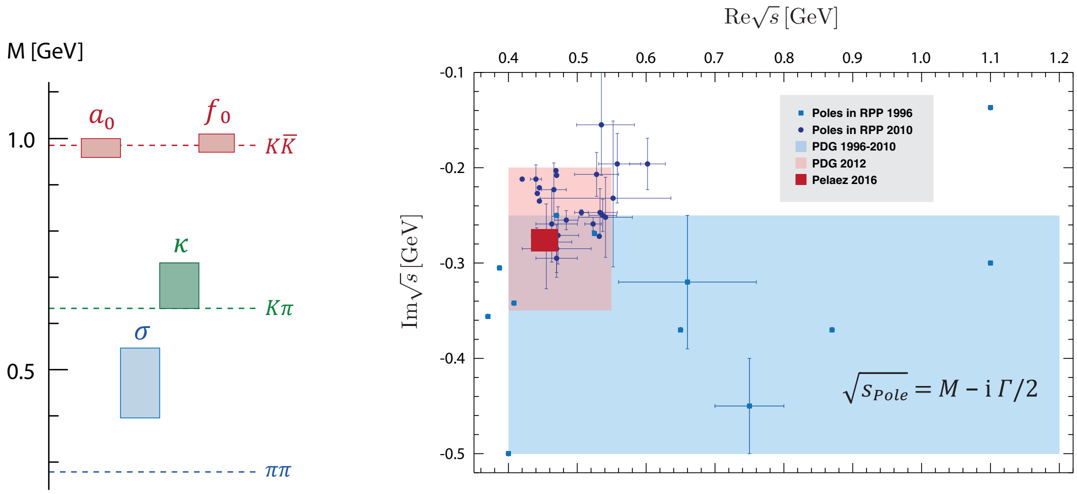


FIG. 3.12: Left: Mass ordering of the light scalar mesons. Right: σ pole location in the complex $\sqrt{s}$ plane (adapted from J. Pelaez, Phys. Rept. 658, 1 (2016)).

Scalar mesons. Another curious case in Fig. 3.10 is the lowest-lying multiplet of scalar 0^{++} mesons. They do not fit into the mass ordering (3.2.45), which would be (roughly) realized if we simply *removed* them from the spectrum. Also the mass ordering inside the multiplet is far from 'ideal': the isosinglet $f_0(500)$ or σ meson is the lightest state, followed by the $K_0^*(700)$ or κ and the almost degenerate $a_0(980)$ and $f_0(980)$. This does not make much sense given the flavor content: why would a_0 and f_0 be mass-degenerate if one is made of light quarks and the other is the $s\bar{s}$ state?

Such arguments initiated the idea that the 0^{++} ground states may not be actual $q\bar{q}$ states but rather **tetraquarks** in the form of diquark-antidiquark combinations.[3] Two quarks can form a diquark through $\mathbf{3}\otimes\mathbf{3} = \overline{\mathbf{3}}\oplus\mathbf{6}$, cf. Fig. 3.9, where it turns out that the color-antitriplet channel $\overline{\mathbf{3}}$ is attractive but the sextet channel $\mathbf{6}$ is repulsive. Hence one can write the combination (3.2.3) also differently:

$$(\mathbf{3}\otimes\mathbf{3})\otimes(\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}) = (\overline{\mathbf{3}}\oplus\mathbf{6})\otimes(\mathbf{3}\oplus\overline{\mathbf{6}}) = (\overline{\mathbf{3}}\otimes\mathbf{3})\oplus\dots = \mathbf{1}\oplus\mathbf{8}\oplus\dots \tag{3.2.48}$$

In flavor space, the antitriplet $\overline{\mathbf{3}}$ corresponds to antisymmetric flavor wave functions $[ud] = ud - du$, $[us]$ and $[ds]$ (up to normalization). Combining a diquark with an antidiquark then produces a singlet and an octet, except with different flavor content: The isoscalar σ is made of light quarks only, whereas both a_0 and f_0 contain $s\bar{s}$ which would make them mass-degenerate. Since the σ and κ lie above the $\pi\pi$ and $K\pi$ thresholds, respectively, they can then simply fall apart without the need for exchanging gluons which would turn them into broad resonances. In fact, the large decay width of the σ has prohibited a precise determination of its pole location until recently (see Fig. 3.12). There has been a long history of support for the non-$q\bar{q}$ nature of the light scalar mesons, although their internal decomposition (diquark-antidiquark, meson-meson, or possible $q\bar{q}$ admixture) is still under debate.[4]

[3] R.L. Jaffe, Phys. Rev. D 15, 267 (1977).

[4] For a review, see J. Pelaez, Phys. Rept. 658, 1 (2016), `arXiv:1510.00653 [hep-ph]`.

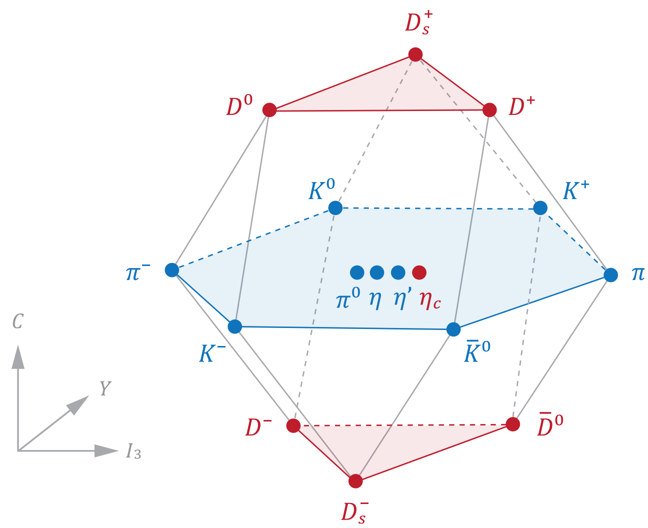


FIG. 3.13: $SU(4)_f$ multiplet arrangement for the pseudoscalar mesons.

Mesons with charm. Let us turn to the spectrum of heavy mesons. In order to include charm quarks, we should start from the group $SU(4)_f$ which has 15 generators. The $SU(4)_f$ symmetry of the Lagrangian is badly broken by the large charm-quark mass, but like in the three-flavor case discussed in Sec. 3.1.1 the diagonal currents corresponding to the three Cartan generators are still conserved. They are related to the quantum numbers I_3, Y and C (charm) which label the states (Fig. 3.13). On top of the light and strange sector, this leads to additional D and D_s mesons:

$$I = \tfrac{1}{2}\,,\; S = 0\,,\; C = \pm 1: \quad \{D^+, D^0, \bar{D}^0, D^-\} = \{c\bar{d}, c\bar{u}, u\bar{c}, d\bar{c}\}\,, \tag{3.2.49}$$

$$I = 0\,,\; S = C = \pm 1: \quad \{D_s^+, D_s^-\} = \{c\bar{s}, s\bar{c}\}. \tag{3.2.50}$$

The separation into different multiplets ($\mathbf{4} \otimes \overline{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15}$) is not useful because due to the broken symmetry the states in the center will mix; ideal mixing amounts to the usual separation into $\frac{1}{\sqrt{2}}(u\bar{u} \pm d\bar{d})$, $s\bar{s}$ and $c\bar{c}$.

Let us focus on the **charmonium** spectrum consisting of $\bar{c}c$ (Fig. 3.14). The first charmonium state to be discovered was the J/ψ, which owes its double name to the simultaneous discovery by two independent collaborations in November 1974 ('November revolution'). The J/ψ and its excitations are vector particles with $J^{PC} = 1^{--}$, so they can be directly produced from a photon in e^+e^- collisions. The naming scheme for the remaining J^{PC} channels is as follows:

$$PC = -+ :\; \eta_c\,, \qquad -- :\; \psi\,, \qquad ++ :\; \chi_c\,, \qquad +- :\; h_c\,. \tag{3.2.51}$$

Since charm quarks are heavy ($m_c \gg \Lambda_{\text{QCD}}$), relativity and chiral symmetry no longer play a major role, which are the two main effects that complicate the light meson spectrum. Thus, effective theories such as **heavy-quark effective theory (HQET)** and **nonrelativistic QCD (NRQCD)** can be used to study heavy-quark physics. In fact, already non-relativistic quark potential models provide an efficient description of the charmonium spectrum.

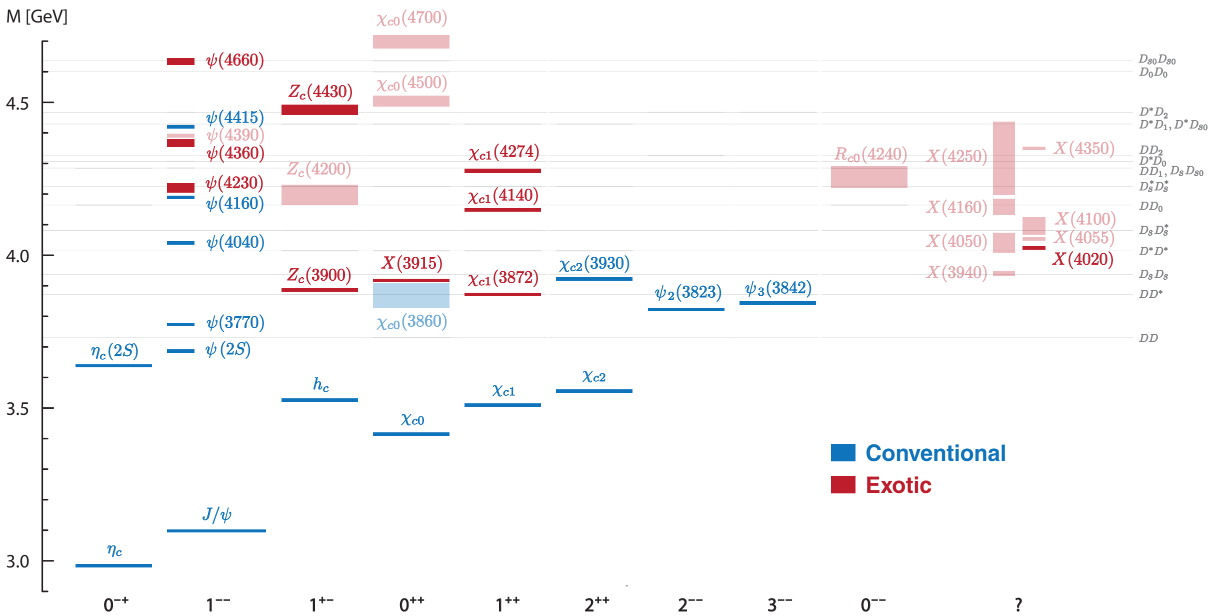


FIG. 3.14: Charmonium spectrum from the PDG 2020 (`https://pdglive.lbl.gov`). Not well-established states are shown in pale colors and the box heights are the mass ranges. The open-charm thresholds are shown in gray.

Exotics. In this sense, the heavy-quark sector is also a much cleaner environment for studying **exotic mesons**. Following a series of discoveries over the past two decades, a number of exotic meson candidates in the charmonium region (the 'XYZ states') are experimentally well-established by now (Fig. 3.14):

- The $\chi_{c1}(3872)$ or $X(3872)$ was first reported by Belle in 2003 and the first exotic charmonium-like state to be found. Its mass is indistinguishable from the $D^0\bar{D}^{*0}$ threshold and it has a very narrow width (< 1.2 MeV).
- The $\psi(4230)$ is one of several exotic candidates in the 1^{--} vector channel, which are produced in e^+e^- collisions.
- The Z_c states with 1^{+-} carry charge and are thus manifestly exotic since their minimal quark content is $c\bar{c}u\bar{d}$, which provides evidence for **tetraquarks**.

The internal structure of four-quark states is under debate. In principle, systems made of $n\bar{n}c\bar{c}$, where n stands for light quarks, could cluster into

- **meson molecules** $(n\bar{c})(\bar{n}c)$ made of two heavy-light mesons, which interact by long-range color-singlet forces such as light meson exchanges;
- compact **diquark-antidiquark** systems $(nc)(\bar{n}\bar{c})$ made of two colored diquarks;
- **hadrocharmonia** $(n\bar{n})(c\bar{c})$ with light mesons that 'orbit' around a heavy core.

Of course, quantum field-theoretically all these configurations can mix together as well as with ordinary $c\bar{c}$ states, but it is conceivable that certain configurations are dominant for particular states, like for example the proximity to a threshold is a typical signal for a molecule.

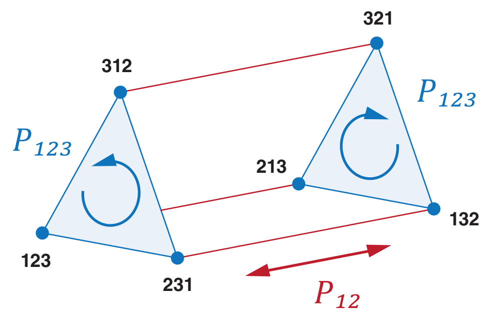


FIG. 3.15: Cayley graph for the permutation group S_3. Any permutation can be reconstructed from a transposition P_{12} and a cyclic permutation P_{123}.

3.2.2 Baryons

Let us come to the baryon sector. Baryons are fermions, so their angular momentum takes half-integer values: $J^P = 1/2^\pm$, $3/2^\pm$, $5/2^\pm$, and so on. If we start again with three flavors u, d and s, then because of

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{3.2.52}$$

in principle each J^P channel can contain $SU(3)$ flavor octets, decuplets and singlets. These are shown in Fig. 3.7, and they can come in the form of ground states and radial excitations. As before, $SU(3)$ flavor breaking entails that baryons with the same I_3 and strangeness S can mix.

The construction of the flavor wave functions is a bit different from the case of mesons since we must combine three quarks instead of a quark and an antiquark. What helps is that baryons satisfy the Pauli principle, i.e., in the flavor-symmetric limit their total (Bethe-Salpeter) wave function

$$\Psi = \text{Dirac} \times \text{Flavor} \times \text{Color} \tag{3.2.53}$$

must be totally antisymmetric under exchange of any two quarks. Here, 'Dirac' is a shorthand for the full spatial and spin (or momentum and spin) contribution that transforms under the Lorentz group, or the rotation group in the non-relativistic case. We can then arrange each part in irreducible representations of the permutation group S_3, with definite symmetry, and figure out in the end which symmetry states are allowed in the combination.

For example concerning the color part, the singlet in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \dots$ is totally antisymmetric (see further below). If we use the quark color basis states $C_1 = R$, $C_2 = G$ and $C_3 = B$, then the color wave function is given by ε_{ijk}:

$$RGB + GBR + BRG - GRB - BGR - RBG = \varepsilon_{ijk}\, C_i\, C_j\, C_k\,. \tag{3.2.54}$$

For the flavor part we must also cast the remaining combinations in Eq. (3.2.52) in permutation-group multiplets, i.e., we must classify them into *simultaneous* irreducible representations of $SU(3)$ and the permutation group S_3.

Permutation group S_3. The permutation group S_3 consists of $3! = 6$ elements. The group manifold can be visualized by the **Cayley graph** in Fig. 3.15: any permutation of an object ψ_{123} can be reconstructed from a transposition P_{12} and a cyclic permutation P_{123}. The former interchanges the indices $1 \leftrightarrow 2$ and the latter is a cyclic permutation $1 \to 2$, $2 \to 3$, $3 \to 1$. The group elements acting on ψ_{123} are given by

$$\begin{array}{ll} 1\,, & P_{12}\,, \\ P_{13}\,P_{12} = P_{123}\,, & P_{23} = P_{12}\,P_{123}\,, \\ P_{23}\,P_{12} = P_{123}^2\,, & P_{13} = P_{12}\,P_{123}^2\,, \end{array} \tag{3.2.55}$$

for example

$$P_{13}\,\psi_{123} = P_{12}\,P_{123}^2\,\psi_{123} = P_{12}\,P_{123}\,\psi_{231} = P_{12}\,\psi_{312} = \psi_{321}\,, \tag{3.2.56}$$

and they are represented by paths along the Cayley graph.

To find the **irreducible representations** of S_3, we define the combinations

$$\psi_1^\pm = \frac{\psi_{123} \pm \psi_{213}}{2}\,, \qquad \psi_2^\pm = \frac{\psi_{231} \pm \psi_{132}}{2}\,, \qquad \psi_3^\pm = \frac{\psi_{312} \pm \psi_{321}}{2}\,. \tag{3.2.57}$$

You can convince yourself that applying P_{12} and P_{123} to them amounts to

$$P_{12}\,\psi_i^\pm = \pm\psi_i^\pm\,, \qquad P_{123}\,\psi_i^\pm = \frac{\psi_j^+ + \psi_j^- \pm (\psi_k^+ - \psi_k^-)}{2}\,, \tag{3.2.58}$$

where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$. If we further define

$$\mathcal{S} = \psi_1^+ + \psi_2^+ + \psi_3^+\,, \qquad \mathcal{A} = \psi_1^- + \psi_2^- + \psi_3^-\,, \tag{3.2.59}$$

then we see that

$$P_{12}\,\mathcal{S} = \mathcal{S}\,, \qquad P_{123}\,\mathcal{S} = \mathcal{S}\,, \qquad P_{12}\,\mathcal{A} = -\mathcal{A}\,, \qquad P_{123}\,\mathcal{A} = \mathcal{A}\,. \tag{3.2.60}$$

Since any permutation can be reconstructed from P_{12} and P_{123}, the combinations $\mathcal{S}$ and $\mathcal{A}$ only transform into themselves, so they form irreducible one-dimensional subspaces under the permutation group. $\mathcal{S}$ is invariant under permutations, so it is a symmetric **singlet**. The antisymmetric singlet (**'antisinglet'**) $\mathcal{A}$ is totally antisymmetric under exchange of any two indices and thus picks up a minus sign under a transposition. The remaining four combinations can be grouped into **doublets**,

$$\mathcal{D}_1 = \begin{bmatrix} \psi_2^- - \psi_3^- \\ \frac{1}{\sqrt{3}}\left(\psi_2^+ + \psi_3^+ - 2\psi_1^+\right) \end{bmatrix}, \qquad \mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}}\left(\psi_2^- + \psi_3^- - 2\psi_1^-\right) \\ \psi_2^+ - \psi_3^+ \end{bmatrix}, \tag{3.2.61}$$

which also transform into themselves and therefore define a two-dimensional subspace:

$$P_{12}\,\mathcal{D}_j = \mathsf{M}_{12}^\mathsf{T}\,\mathcal{D}_j\,, \qquad P_{123}\,\mathcal{D}_j = \mathsf{M}_{123}^\mathsf{T}\,\mathcal{D}_j\,. \tag{3.2.62}$$

The representation matrices (M^T denotes the matrix transpose) are given by

$$\mathsf{M}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{M}_{123} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \tag{3.2.63}$$

from where all other ones can be reconstructed through Eq. (3.2.55), e.g.:

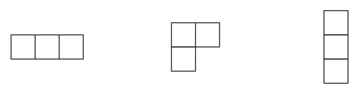
$$\begin{aligned} P_{23}\,\mathcal{D}_j &= P_{12}\,P_{123}\,\mathcal{D}_j = P_{12}\left(\mathsf{M}_{123}^\mathsf{T}\,\mathcal{D}_j\right) = \mathsf{M}_{123}^\mathsf{T}\,P_{12}\,\mathcal{D}_j \\ &= \mathsf{M}_{123}^\mathsf{T}\,\mathsf{M}_{12}^\mathsf{T}\,\mathcal{D}_j = (\mathsf{M}_{12}\,\mathsf{M}_{123})^\mathsf{T}\,\mathcal{D}_j\,. \end{aligned} \tag{3.2.64}$$

The upper (lower) components of the doublets are antisymmetric (symmetric) under transpositions P_{12} and we denote them by

$$\mathcal{D}_j = \begin{bmatrix} a_j \\ s_j \end{bmatrix}. \tag{3.2.65}$$

In the language of Young diagrams (see Appendix A.3), the irreducible representations of S_3 correspond to

$\mathcal{S}$ $\mathcal{D}_j$ $\mathcal{A}$



In practice we will also need the tensor products of S_3 multiplets. Given two sets of singlets $\mathcal{S}$, $\mathcal{S}'$, antisinglets $\mathcal{A}$, $\mathcal{A}'$ and doublets $\mathcal{D} = [a, s]$, $\mathcal{D}' = [a', s']$, there are 16 possible combinations

$$\{\mathcal{S},\, \mathcal{A},\, a,\, s\} \times \{\mathcal{S}',\, \mathcal{A}',\, a',\, s'\} \tag{3.2.66}$$

which we can again arrange into multiplets. Clearly, the products of two singlets ($\mathcal{S}\,\mathcal{S}'$) or antisinglets ($\mathcal{A}\,\mathcal{A}'$) must be singlets. The inner product $\mathcal{D}\cdot\mathcal{D}'$ of two doublets is also a singlet and invariant under any permutation because the representation matrices $\mathsf{M} \in \{\mathsf{M}_{12},\, \mathsf{M}_{123}\}$ are orthogonal:

$$(\mathsf{M}^\mathsf{T}\mathcal{D})\cdot(\mathsf{M}^\mathsf{T}\,\mathcal{D}') = \mathcal{D}_k\,(\mathsf{M}\mathsf{M}^\mathsf{T})_{kl}\,\mathcal{D}'_l = \mathcal{D}\cdot\mathcal{D}'\,. \tag{3.2.67}$$

Therefore, there are three possibilities for constructing singlets in the product space:

$$\mathcal{S}\,\mathcal{S}', \qquad \mathcal{A}\,\mathcal{A}', \qquad \mathcal{D}\cdot\mathcal{D}' = aa' + ss'\,. \tag{3.2.68}$$

Antisinglets are obtained from

$$\mathcal{S}\,\mathcal{A}', \qquad \mathcal{A}\,\mathcal{S}', \qquad \mathcal{D}\wedge\mathcal{D}' := as' - sa'\,, \tag{3.2.69}$$

where we defined an antisymmetric wedge product, and doublets are formed by

$$\begin{matrix} \mathcal{S}\,\mathcal{D}', & \quad \mathcal{A}\,(\varepsilon\,\mathcal{D}')\,, \\ \mathcal{S}'\mathcal{D}, & \quad \mathcal{A}'\,(\varepsilon\,\mathcal{D})\,, \end{matrix} \qquad \mathcal{D}*\mathcal{D}' := \begin{bmatrix} as' + sa' \\ aa' - ss' \end{bmatrix}, \tag{3.2.70}$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad\Rightarrow\quad \varepsilon\,\mathcal{D} = \begin{bmatrix} s \\ -a \end{bmatrix}. \tag{3.2.71}$$

This covers all 16 possibilities. You can easily check this using Eqs. (3.2.60) and (3.2.62) written down for a, s, a' and s': the singlets stay invariant under permutations, the antisinglets pick up a minus sign for odd permutations, and the doublets transform under M_{12} and M_{123}.

	uuu	uud	ddu	ddd	uus	uds	dds	ssu	ssd	sss
$\mathcal{S}$	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	Ω^-
$\mathcal{D}_1$		p	n		Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	
$\mathcal{D}_2$						Λ^0				
$\mathcal{A}$						Λ^0				

TABLE 3.3: $SU(3)_f$ flavor wave functions for baryons.

Flavor wave functions for baryons. Suppose u, d and s denote flavor vectors that transform under the fundamental representation of $SU(3)$, for example in a Cartesian basis. Combining three of them gives $3\times3\times3=27$ possible combinations, which would transform under the 27-dimensional reducible representation of $SU(3)$. The irreducible representations contained in $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}$ differ by their symmetry, so we must find the combined irreducible representations of $SU(3)$ and S_3.

To construct flavor wave functions from the tensor products of three flavor vectors, e.g. for a baryon with flavor content uud such as the proton or the Δ^+, we write

$$\begin{aligned}
\psi_{123} &= u_i u_j d_k = (uud)_{ijk}\,, & \psi_{213} &= u_j u_i d_k = (uud)_{ijk}\,,\\
\psi_{231} &= u_j u_k d_i = (duu)_{ijk}\,, & \psi_{132} &= u_i u_k d_j = (udu)_{ijk}\,,\\
\psi_{312} &= u_k u_i d_j = (udu)_{ijk}\,, & \psi_{321} &= u_k u_j d_i = (duu)_{ijk}\,.
\end{aligned}\tag{3.2.72}$$

The combinations in Eq. (3.2.57) become

$$\psi_1^+ = uud\,,\qquad \psi_1^- = 0\,,\qquad \psi_2^\pm = \pm\psi_3^\pm = \pm\frac{ud\pm du}{2}\,u\,,\tag{3.2.73}$$

so we arrive at the multiplets

$$\begin{aligned}
\mathcal{S}(uud) &= uud + udu + duu\,, & \mathcal{D}_1(uud) &= \begin{bmatrix} duu - udu \\ \frac{1}{\sqrt{3}}\,(udu+duu-2uud)\end{bmatrix},\\
\mathcal{A}(uud) &= 0\,, & \mathcal{D}_2(uud) &= 0\,.
\end{aligned}\tag{3.2.74}$$

Apart from overall normalization, $\mathcal{S}(uud)$ is the flavor wave function of the Δ^+ and $\mathcal{D}_1(uud)$ is that of the proton. Had we started from ddu instead of uud, we would have obtained the wave functions for the Δ^0 and the neutron (replace $u\leftrightarrow d$ in the equation above). The combination uuu returns only a singlet (Δ^{++}), and from uds we get everything: $\mathcal{S}$, $\mathcal{A}$ and two doublets.

If we take all 10 combinations with different flavor content into account (uuu, ddd, sss, uud, uus, ddu, dds, ssu, ssd, uds), the permutation group gives us

- 10 singlets, which form the flavor decuplet with Δ, Σ, Ξ and Ω,
- 8 doublets which form the flavor octet, including proton, neutron, Σ, Ξ and Λ,
- and one antisinglet from uds, the flavor singlet for Λ.

These are just the irreducible representations of $SU(3)_f$: decuplet, octet and singlet. The resulting states are collected in Table 3.3 and written explicitly in Table 3.4.

		I	I_3	S	
uud	p	$1/2$	$1/2$	0	$\frac{1}{\sqrt{2}}\begin{bmatrix} udu - duu \\ -\frac{1}{\sqrt{3}}(udu + duu - 2uud) \end{bmatrix}$
udd	n	$1/2$	$-1/2$	0	$\frac{1}{\sqrt{2}}\begin{bmatrix} udd - dud \\ \frac{1}{\sqrt{3}}(dud + udd - 2ddu) \end{bmatrix}$
uus	Σ^+	1	1	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix} usu - suu \\ -\frac{1}{\sqrt{3}}(usu + suu - 2uus) \end{bmatrix}$
uds	Σ^0	1	0	-1	$\frac{1}{2}\begin{bmatrix} sud - usd + sdu - dsu \\ \frac{1}{\sqrt{3}}(sud + usd + sdu + dsu - 2uds - 2dus) \end{bmatrix}$
dds	Σ^-	1	-1	-1	$\frac{1}{\sqrt{2}}\begin{bmatrix} dsd - sdd \\ -\frac{1}{\sqrt{3}}(dsd + sdd - 2dds) \end{bmatrix}$
uds	Λ^0	0	0	-1	$\frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{3}}(2uds - 2dus + usd - dsu + sdu - sud) \\ usd - dsu + sud - sdu \end{bmatrix}$
uss	Ξ^0	$1/2$	$1/2$	-2	$\frac{1}{\sqrt{2}}\begin{bmatrix} uss - sus \\ \frac{1}{\sqrt{3}}(sus + uss - 2ssu) \end{bmatrix}$
dss	Ξ^-	$1/2$	$-1/2$	-2	$\frac{1}{\sqrt{2}}\begin{bmatrix} dss - sds \\ \frac{1}{\sqrt{3}}(sds + dss - 2ssd) \end{bmatrix}$
uuu	Δ^{++}	$3/2$	$3/2$	0	uuu
uud	Δ^+	$3/2$	$1/2$	0	$\frac{1}{\sqrt{3}}(uud + udu + duu)$
udd	Δ^0	$3/2$	$-1/2$	0	$\frac{1}{\sqrt{3}}(udd + dud + ddu)$
ddd	Δ^-	$3/2$	$3/2$	0	ddd
uus	Σ^+	1	1	-1	$\frac{1}{\sqrt{3}}(uus + usu + suu)$
uds	Σ^0	1	0	-1	$\frac{1}{\sqrt{6}}(uds + sud + dsu + dus + usd + sdu)$
dds	Σ^-	1	-1	-1	$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$
uss	Ξ^0	$1/2$	$1/2$	-2	$\frac{1}{\sqrt{3}}(uss + sus + ssu)$
dss	Ξ^-	$1/2$	$-1/2$	-2	$\frac{1}{\sqrt{3}}(dss + sds + ssd)$
sss	Ω^-	0	0	-3	sss
uds	Λ^0	0	0	-1	$\frac{1}{\sqrt{6}}(uds + sud + dsu - dus - usd - sdu)$

TABLE 3.4: $SU(3)_f$ flavor wave functions for octet, decuplet and singlet baryons.

Including charm as a fourth flavor, we can immediately extend the construction to $SU(4)_f$ which would give us 20 singlets, 20 doublets and 4 antisinglets:

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{20}_S \oplus \mathbf{20}_{M_A} \oplus \mathbf{20}_{M_S} \oplus \mathbf{4}_A\,. \tag{3.2.75}$$

In the $SU(2)_f$ case, on the other hand, we get four singlets (the four Δ baryons) and two doublets (proton and neutron):

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_S \oplus \mathbf{2}_{M_A} \oplus \mathbf{2}_{M_S}\,. \tag{3.2.76}$$

If we identified $SU(2)$ with spin instead of flavor, this would give us the three-spinor wave functions, e.g. the four symmetric ones:

$$|{\uparrow\uparrow\uparrow}\rangle, \quad \tfrac{1}{\sqrt{3}}\left(|{\uparrow\uparrow\downarrow}\rangle + |{\uparrow\downarrow\uparrow}\rangle + |{\downarrow\uparrow\uparrow}\rangle\right), \quad \tfrac{1}{\sqrt{3}}\left(|{\downarrow\downarrow\uparrow}\rangle + |{\downarrow\uparrow\downarrow}\rangle + |{\uparrow\downarrow\downarrow}\rangle\right), \quad |{\downarrow\downarrow\downarrow}\rangle. \tag{3.2.77}$$

Full baryon wave function. The remaining question is what the Dirac part in Eq. (3.2.53) looks like. From the above analysis we conclude that even without knowing its explicit form, we can also arrange it into permutation group multiplets $\mathcal{S}$, $\mathcal{A}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ to write

$$\Psi = \{\mathcal{S},\, \mathcal{A},\, \mathcal{D}_1,\, \mathcal{D}_2\}_D \times \{\mathcal{S},\, \mathcal{A},\, \mathcal{D}_1,\, \mathcal{D}_2\}_F \times \mathcal{A}_C \stackrel{!}{=} \mathcal{A}_{\text{total}}\,. \tag{3.2.78}$$

Because color is antisymmetric, the Dirac-flavor part must be symmetric. This leaves the three possible combinations in Eq. (3.2.68):

$$\mathcal{A}_{\text{total}} \;=\; \begin{cases} (\mathcal{D}_D \cdot \mathcal{D}_F)\,\mathcal{A}_C & \text{(octet)}\,, \\ (\mathcal{S}_D\,\mathcal{S}_F)\,\mathcal{A}_C & \text{(decuplet)}\,, \\ (\mathcal{A}_D\,\mathcal{A}_F)\,\mathcal{A}_C & \text{(singlet)}\,. \end{cases} \tag{3.2.79}$$

That is, flavor octet baryons come with a mixed-symmetric Dirac part, decuplet baryons with a symmetric and flavor-singlet baryons with an antisymmetric Dirac part.

In principle the Dirac part can be constructed from the Bethe-Salpeter wave function

$$\langle 0|\, \mathsf{T}\psi_\alpha(x_1)\,\psi_\beta(x_2)\,\psi_\gamma(x_3)\,|\lambda\rangle \tag{3.2.80}$$

in analogy to Eqs. (3.1.136–3.1.137): In momentum space it has the structure

$$\Psi_{\alpha\beta\gamma}(p_1,p_2,p_3) = \sum_{i=1}^{N} f_i(q_1^2,\, q_2^2,\, q_1\cdot p,\, q_2\cdot p,\, q_1\cdot q_2,\, p^2 = m_\lambda^2)\,\tau_i(p_1,p_2,p_3)_{\alpha\beta\gamma}\,, \tag{3.2.81}$$

where p is the onshell momentum of the baryon, q_1 and q_2 are the two relative momenta in the system, and the τ_i form a linearly independent and complete tensor basis. These can be grouped into S_3 multiplets, so that the symmetry properties are inherited by the dressing functions f_i.

It is instructive to go back to the **nonrelativistic quark model**, like we did in the discussion of mesons below Eq. (3.2.44). In that case J^P, L and S are good quantum numbers. The Dirac parts are taken to be the direct product of $O(3)$ spatial and $SU(2)$ spin wave functions. The combination of three spins $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$ in a three-quark qqq state only permits total spin $S = \frac{1}{2}$ or $S = \frac{3}{2}$. The corresponding wave functions are those in Eq. (3.2.77), i.e., there are four permutation-group singlets $\mathcal{S}_S$ (subscript S for spin) with $S = 3/2$ and two doublets $\mathcal{D}_S$ with spin $S = 1/2$.

The $SU(2)$ spin states ($\mathcal{D}_S$, $\mathcal{S}_S$) are then combined with the $SU(3)$ flavor states ($\mathcal{D}_F$, $\mathcal{S}_F$, $\mathcal{A}_F$) into spin-flavor multiplets according to Eqs. (3.2.68–3.2.70):

- **56-plet** $\mathcal{S}_{SF}$: $\mathcal{D}_S \cdot \mathcal{D}_F \to 2 \times 8 = 16$ spin-flavor states,
  $\mathcal{S}_S\, \mathcal{S}_F \to 4 \times 10 = 40$,
- **70-plet** $\mathcal{D}_{SF}$: $\mathcal{D}_S * \mathcal{D}_F \to 2 \times 8 = 16$,
  $\mathcal{D}_S\, \mathcal{S}_F \to 2 \times 10 = 20$,
  $(\varepsilon\, \mathcal{D}_S)\, \mathcal{A}_F \to 2 \times 1 = 2$,
  $\mathcal{S}_S\, \mathcal{D}_F \to 4 \times 8 = 32$,
- **20-plet** $\mathcal{A}_{SF}$: $\mathcal{D}_S \wedge \mathcal{D}_F \to 2 \times 8 = 16$,
  $\mathcal{S}_S\, \mathcal{A}_F \to 4 \times 1 = 4$.

This is the '$SU(6)$-symmetric' quark model classification, since $SU(2)_{\text{spin}} \times SU(3)_{\text{flavor}} \sim SU(6)_{\text{spin-flavor}}$ and in $SU(6)$ one has

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_{M_A} \oplus \mathbf{70}_{M_S} \oplus \mathbf{20}_A \,. \tag{3.2.82}$$

Keep in mind, however, that different spin polarizations do not correspond to different particles: only the number of states in a flavor multiplet counts the number of baryons we expect to find in the spectrum.

The **spatial wave functions** can depend on the total coordinate $\boldsymbol{R}$ and the relative (Jacobi) coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$:

$$\boldsymbol{R} = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3}{\sqrt{3}}\,, \quad \boldsymbol{\rho} = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\sqrt{2}}\,, \quad \boldsymbol{\lambda} = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 - 2\boldsymbol{x}_3}{\sqrt{6}}\,. \tag{3.2.83}$$

After removing the center-of-mass motion induced by $\boldsymbol{R}$, the spatial wave functions $\phi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ only depend on the relative coordinates. These can be arranged in a permutation-group doublet, since from Eq. (3.2.62) one can verify

$$P_{12} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\lambda} \end{bmatrix} = \mathsf{M}_{12}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\lambda} \end{bmatrix}, \qquad P_{123} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\lambda} \end{bmatrix} = \mathsf{M}_{123}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\lambda} \end{bmatrix}. \tag{3.2.84}$$

From a doublet $\mathcal{D}$ one can construct further multiplets such as the $O(3)$ invariants

$$\mathcal{S} = \mathcal{D} \cdot \mathcal{D} = \boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2\,, \qquad \mathcal{D}' = \mathcal{D} * \mathcal{D} = \begin{bmatrix} 2\boldsymbol{\rho} \cdot \boldsymbol{\lambda} \\ \boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2 \end{bmatrix}, \tag{3.2.85}$$

and in this way also the spatial wave function $\phi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ can produce permutation-group singlets, doublets and antisinglets.

The spatial wave functions are usually set up in a spherical harmonic oscillator basis

$$\phi_L(\boldsymbol{\rho},\boldsymbol{\lambda}) = \sum_{n_\rho,l_\rho,n_\lambda,l_\lambda} c^L_{n_\rho l_\rho n_\lambda l_\lambda}\,\left[\phi_{n_\rho l_\rho}(\boldsymbol{\rho})\otimes\phi_{n_\lambda l_\lambda}(\boldsymbol{\lambda})\right]_L\,, \tag{3.2.86}$$

where both internal motions support radial ($n_\alpha > 0$) and orbital ($l_\alpha > 0$) excitations. With $n = n_\rho + n_\lambda$ and $l = l_\rho + l_\lambda$, the total orbital angular momentum is constructed from $L = |l_\rho - l_\lambda| \dots l_\rho + l_\lambda$ and the parity of the state is $P = (-1)^l$. This yields excitations bands for the 'band quantum number' $N = 2n + l$ corresponding to the same energy. The resulting spatial wave functions can be arranged into permutation-group multiplets $\mathcal{S}_O$ (subscript O for orbital), $\mathcal{D}_O$ and $\mathcal{A}_O$, which are finally combined with the spin-flavor wave functions to yield the totally symmetric combinations

$$\begin{aligned}
\mathcal{S}_O\,\mathcal{S}_{SF} \quad &\rightarrow \quad \mathcal{S}_O\,\mathcal{D}_S\cdot\mathcal{D}_F\,,\;\mathcal{S}_O\,\mathcal{S}_S\,\mathcal{S}_F\,,\\
\mathcal{D}_O\cdot\mathcal{D}_{SF} \quad &\rightarrow \quad \mathcal{D}_O\cdot(\mathcal{D}_S*\mathcal{D}_F)\,,\;\mathcal{D}_O\cdot(\mathcal{D}_S\,\mathcal{S}_F)\,,\;\mathcal{D}_O\cdot(\varepsilon\,\mathcal{D}_S)\,\mathcal{A}_F\,,\;\mathcal{D}_O\cdot(\mathcal{S}_S\,\mathcal{D}_F)\,,\\
\mathcal{A}_O\,\mathcal{A}_{SF} \quad &\rightarrow \quad \mathcal{A}_O\,(\mathcal{D}_S\wedge\mathcal{D}_F)\,,\;\mathcal{A}_O\,\mathcal{S}_S\,\mathcal{A}_F\,.
\end{aligned}$$

Since the spatial wave functions carry definite L and P and the spin wave functions definite S, their combination $J = |L-S| \dots L+S$ determines J^P. The resulting states and their flavor assignments are listed in Table 3.5.

One can see that the **ground states** ($N = 0$) correspond to flavor octet baryons with $J = \frac{1}{2}^+$ and decuplet baryons with $\frac{3}{2}^+$. We could have inferred this directly from Eq. (3.2.79): For ground states the orbital wave functions are spatially symmetric, i.e., permutation-group singlets, so the different Dirac multiplets can only come from the spin. Ground states have $L = 0$ and thus $J = S$, so the only possible combinations are

$$\mathcal{A}_{\text{total}} \sim \begin{cases} (\mathcal{D}_S\cdot\mathcal{D}_F)\,\mathcal{A}_C & (J = \frac{1}{2}^+,\ \text{octet})\,,\\ \quad(\mathcal{S}_S\,\mathcal{S}_F)\,\mathcal{A}_C & (J = \frac{3}{2}^+,\ \text{decuplet})\,. \end{cases} \tag{3.2.87}$$

Because there is no antisymmetric spin wave function $\mathcal{A}_S$, the flavor-singlet baryons Λ^0 cannot appear as ground states.

What Table 3.5 also shows is that the quark model predicts *a lot* of states. While the 'bands' for $N = 0$ and $N = 1$ can be identified with experimentally known baryons, already the $N = 2$ and especially the $N = 3$ states have not all been observed. This is the so-called **missing resonances problem**, which could have several explanations:

- We simply have not found them yet. Excited baryons (generically called N^*) have traditionally been extracted from $N\pi$ scattering ($N\pi \to N\pi$), but if they did not strongly couple to $N\pi$ it would be hard to see their peaks in experimental cross sections (remember Eq. (3.1.121)). Recent photoproduction experiments (e.g. $\gamma N \to N\pi$) have indeed found new states, but the spectrum as of today is still quite sparse compared to what the quark model predicts.
- If two quarks inside a baryon clustered to a **diquark**, this would freeze internal excitation degrees of freedom and we should see fewer states in the spectrum.
- The assumptions we made (nonrelativistic quark model, harmonic oscillator) are simply too drastic to provide a realistic description of light baryons.

N	L^P	O	SF	S	F	J^P
0	0^+	$\mathcal{S}_0$	**56**	$\frac{1}{2}$	**8**	$\frac{1}{2}^+$
				$\frac{3}{2}$	**10**	$\frac{3}{2}^+$
1	1^-	$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**10**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**1**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{3}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
2	0^+	$\mathcal{S}_0$	**56**	$\frac{1}{2}$	**8**	$\frac{1}{2}^+$
				$\frac{3}{2}$	**10**	$\frac{3}{2}^+$
		$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{1}{2}^+$
				$\frac{1}{2}$	**10**	$\frac{1}{2}^+$
				$\frac{1}{2}$	**1**	$\frac{1}{2}^+$
				$\frac{3}{2}$	**8**	$\frac{3}{2}^+$
	1^+	$\mathcal{A}_0$	**20**	$\frac{1}{2}$	**8**	$\frac{1}{2}^+, \frac{3}{2}^+$
				$\frac{3}{2}$	**1**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$
	2^+	$\mathcal{S}_0$	**56**	$\frac{1}{2}$	**8**	$\frac{3}{2}^+, \frac{5}{2}^+$
				$\frac{3}{2}$	**10**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$
		$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{3}{2}^+, \frac{5}{2}^+$
				$\frac{1}{2}$	**10**	$\frac{3}{2}^+, \frac{5}{2}^+$
				$\frac{1}{2}$	**1**	$\frac{3}{2}^+, \frac{5}{2}^+$
				$\frac{3}{2}$	**8**	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+, \frac{7}{2}^+$
3	1^-	$\mathcal{S}_0$	**56**	$\frac{1}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{3}{2}$	**10**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
		$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**10**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**1**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{3}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
		$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**10**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{1}{2}$	**1**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{3}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
		$\mathcal{A}_0$	**20**	$\frac{1}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-$
				$\frac{3}{2}$	**1**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$
	2^-	$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{3}{2}^-, \frac{5}{2}^-$
				$\frac{1}{2}$	**10**	$\frac{3}{2}^-, \frac{5}{2}^-$
				$\frac{1}{2}$	**1**	$\frac{3}{2}^-, \frac{5}{2}^-$
				$\frac{3}{2}$	**8**	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-$
	3^-	$\mathcal{S}_0$	**56**	$\frac{1}{2}$	**8**	$\frac{5}{2}^-, \frac{7}{2}^-$
				$\frac{3}{2}$	**10**	$\frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-, \frac{9}{2}^-$
		$\mathcal{D}_0$	**70**	$\frac{1}{2}$	**8**	$\frac{5}{2}^-, \frac{7}{2}^-$
				$\frac{1}{2}$	**10**	$\frac{5}{2}^-, \frac{7}{2}^-$
				$\frac{1}{2}$	**1**	$\frac{5}{2}^-, \frac{7}{2}^-$
				$\frac{3}{2}$	**8**	$\frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-, \frac{9}{2}^-$
		$\mathcal{A}_0$	**20**	$\frac{1}{2}$	**8**	$\frac{5}{2}^-, \frac{7}{2}^-$
				$\frac{3}{2}$	**1**	$\frac{3}{2}^-, \frac{5}{2}^-, \frac{7}{2}^-, \frac{9}{2}^-$

TABLE 3.5: Quark-model classification of light and strange baryons up to $N \leq 3$.

It is also amusing to think about a world without color. The Δ^{++} carries three up quarks (uuu) and has all spins aligned ($\uparrow\uparrow\uparrow$), which does not yield a totally antisymmetric wave function — which was historically one of the motivations for introducing the color degree of freedom. If we wanted to respect the Pauli principle *without* color, then Eq. (3.2.69) provides us with the following options:

$$\mathcal{A}_{\text{total}} = \begin{cases} \mathcal{D}_D \wedge \mathcal{D}_F & \text{(octet)}\,, \\ \mathcal{A}_D\,\mathcal{S}_F & \text{(decuplet)}\,, \\ \mathcal{S}_D\,\mathcal{A}_F & \text{(singlet)}\,. \end{cases} \tag{3.2.88}$$

With the $SU(2)$ spin wave functions $\mathcal{D}_S$ and $\mathcal{S}_S$ (but no $\mathcal{A}_S$) we could still construct nucleons but not Δ baryons, at least not as ground states.

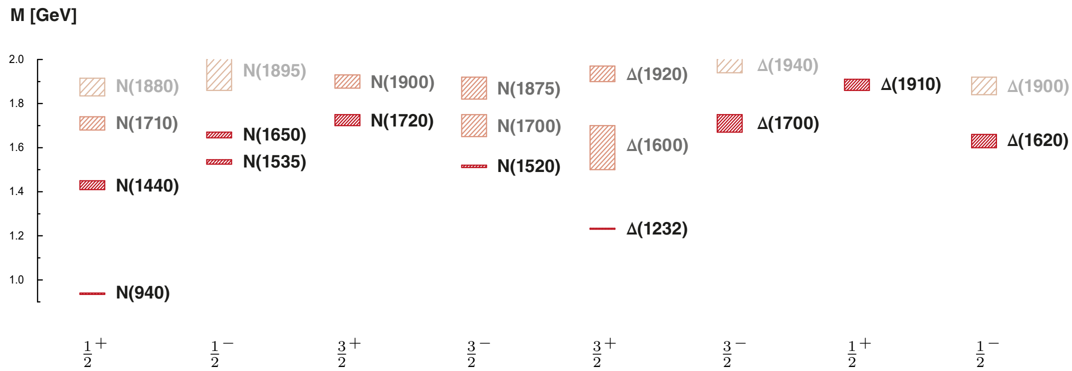


FIG. 3.16: Light baryon spectrum for $J^P = \frac{1}{2}^{\pm}$ and $\frac{3}{2}^{\pm}$ from the PDG.

Light baryons. Let us have a look at the experimental spectrum of light and strange baryons (Table 3.6). In contrast to mesons, the naming scheme is the same for different J^P channels, i.e., all states with $I = \frac{1}{2}$ and $S = 0$ are called nucleons, all states with $I = \frac{3}{2}$ and $S = 0$ are called Δ baryons, etc. From the point of view of the Poincaré group, each J^P channel contains one 'ground state' plus radial excitations. Due to $SU(3)_V$ breaking, multiplets with the same I_3 and S can mix. This affects the baryons containing strange quarks (the **hyperons**): the Λ states (uds) can be mixtures of **8** and **1** and the Σ and Ξ states can be mixtures of **8** and **10**.

The well established states are the ground states that are also predicted by the quark model: the octet baryons with $J^P = \frac{1}{2}^+$ and the decuplet baryons with $J^P = \frac{3}{2}^+$. Their lightest members are the nucleon (proton and neutron) and the $\mathbf{\Delta(1232)}$ resonance. Since they carry different three-quark spin S (see Table 3.5), their mass difference of about 300 MeV can be understood as a hyperfine splitting due to spin-dependent interactions. The Δ resonance decays almost exclusively into $N\pi$ and thus appears as a prominent peak in $N\pi$ scattering.

The $N = 1$ band in Table 3.5 can still be identified with experimental states, e.g. in the nucleon channel the $(\frac{1}{2}, \mathbf{8})$ states would correspond to the $N(1535)$ and $N(1520)$, where the former is the parity partner of the nucleon (see also Fig. 3.16). For the higher-lying states the quark-model identification becomes more problematic: the $N = 2$ band already overpredicts the positive-parity spectrum for $\frac{3}{2}^+$ states, and the $N = 3$ band contains over 20 negative-parity states which have not been seen in experiments.

An open question concerns the **Roper** resonance $N(1440)$, which is the first radial excitation of the nucleon but has properties that are incompatible with the quark model; for example, its mass is lower than that of the $N(1535)$. The Roper has been suggested to be a **dynamically generated resonance**, in the sense that the interactions between nucleons and pions could generate additional states on top of qqq configurations. This ties in with the **'meson cloud'** picture, where baryons are thought to be surrounded by clouds of light pseudoscalar mesons which change their properties. A similar case is the $\Lambda(1405)$ with $J^P = \frac{1}{2}^-$ which is also not well described by quark models. From a microscopic point of view, such effects would signal a multiquark admixture for light baryons similarly to the meson spectrum. For these reasons, a thorough understanding of light baryons from QCD remains an open problem.

M	I	S	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{5}{2}^+$	$\frac{7}{2}^+$	$\frac{9}{2}^+$	$\frac{11}{2}^+$	$\frac{13}{2}^+$
8	$\frac{1}{2}$	0	**N (939)** N (1440) N (1710) N (1880) N (2100) N (2300)	N (1720) N (1900)	N (1680) N (1860) N (2000)	N (1990)	N (2220)		N (2700)
10	$\frac{3}{2}$	0	Δ (1910)	**$\boldsymbol{\Delta}$ (1232)** Δ (1600) Δ (1920)	Δ (1905) Δ (2000)	Δ (1950)	Δ (2300)	Δ (2420)	
8, 1	0	-1	**$\boldsymbol{\Lambda}$ (1116)** Λ (1600) Λ (1810)	Λ (1890)	Λ (1820) Λ (2110)	Λ (2085)	Λ (2350)		
8, 10	1	-1	**$\boldsymbol{\Sigma}$ (1193)** Σ (1660) Σ (1880)	**$\boldsymbol{\Sigma}$ (1385)**	Σ (1915)	Σ (2030)			
8, 10	$\frac{1}{2}$	-2	**$\boldsymbol{\Xi}$ (1318)**	**$\boldsymbol{\Xi}$ (1530)**					
10	0	-3		**$\boldsymbol{\Omega}$ (1672)**					

M	I	S	$\frac{1}{2}^-$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{7}{2}^-$	$\frac{9}{2}^-$	$\frac{11}{2}^-$	$\frac{13}{2}^-$
8	$\frac{1}{2}$	0	N (1535) N (1650) N (1895)	N (1520) N (1700) N (1875) N (2120)	N (1675) N (2060) N (2570)	N (2190)	N (2250)	N (2600)	
10	$\frac{3}{2}$	0	Δ (1620) Δ (1900)	Δ (1700) Δ (1940)	Δ (1930)	Δ (2200)	Δ (2400)		Δ (2750)
8, 1	0	-1	Λ (1380) Λ (1405) Λ (1670) Λ (1800)	Λ (1520) Λ (1690)	Λ (1830)	Λ (2100)			
8, 10	1	-1	Σ (1750) Σ (1900)	Σ (1670) Σ (1910)	Σ (1775)				
8, 10	$\frac{1}{2}$	-2	Ξ (1690)	Ξ (1820)					
10	0	-3							

TABLE 3.6: Light and strange baryon spectrum in terms of J^P, isospin I and strangeness S from the PDG 2020 (https://pdglive.lbl.gov). Only established states (two-, three- and four-star resonances) are included. The ground states are shown in color.

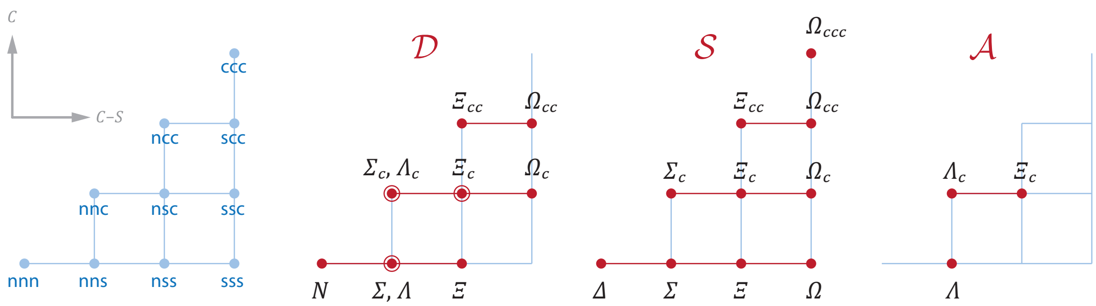


FIG. 3.17: Charmed baryon multiplets in the $\{C-S, C\}$ plane. The left figure shows the quark content for each state, where n stands for light quarks.

	uuc	udc	ddc	usc	dsc	ssc	ucc	dcc	scc	ccc
$\mathcal{S}$	Σ_c^{++}	Σ_c^{+}	Σ_c^{0}	Ξ_c^{+}	Ξ_c^{0}	Ω_c^{0}	Ξ_{cc}^{++}	Ξ_{cc}^{+}	Ω_{cc}^{+}	Ω_{ccc}^{+}
$\mathcal{D}_1$	Σ_c^{++}	Σ_c^{+}	Σ_c^{0}	Ξ_c^{+}	Ξ_c^{0}	Ω_c^{0}	Ξ_{cc}^{++}	Ξ_{cc}^{+}	Ω_{cc}^{+}	
$\mathcal{D}_2$		Λ_c^{+}		Ξ_c^{+}	Ξ_c^{0}					
$\mathcal{A}$		Λ_c^{+}		Ξ_c^{+}	Ξ_c^{0}					

TABLE 3.7: $SU(3)_f$ flavor wave functions for baryons.

Charmed baryons. The extension of Eqs. (3.2.72–3.2.74) to construct the flavor wave functions of charmed baryons is straightforward: start with a given quark content like uuc and work out the multiplets. This produces new singlets, doublets and antisinglets, which are collected in Table 3.7 and add to the former ones to yield

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{20}_S \oplus \mathbf{20}_{M_A} \oplus \mathbf{20}_{M_S} \oplus \mathbf{4}_A \,. \tag{3.2.89}$$

The resulting flavor multiplets are shown in Fig. 3.17 and contain the $SU(3)$ octet, decuplet and singlet as their bottom levels. As before, because the $SU(4)_f$ symmetry is badly broken, states with the same quark content (the same I_3, strangeness S and number of charm quarks C) will mix.

For singly-charmed baryons, the multiplet partners of the $J^P = \frac{1}{2}^+$ octet and $\frac{3}{2}^+$ decuplet baryons are experimentally established, along with a few other states with different J^P and some whose quantum numbers have not yet been determined. So far there is evidence for only one doubly charmed Ξ_{cc}^{++} baryon; presumably these would have a very different structure from light baryons and resemble a heavy 'double-star' system with an attached light 'planet'.

Pentaquarks? Another type of baryon made of charm quarks was recently observed by the LHCb collaboration, who found several peaks in the $J/\psi p$ spectrum in the $4300 \dots 4500$ MeV region. Since this implies a minimal quark content $uudc\bar{c}$, it would be the first experimental evidence for **pentaquarks**. The proximity of those peaks to the $\Sigma_c \bar{D}$ and $\Sigma_c \bar{D}^*$ thresholds suggests a molecular explanation in terms of meson-baryon molecules, in analogy to exotic meson candidates in the charmonium sector.

References and further reading

Symmetries and Ward-Takahashi identities:

- M. Maggiore, *A Modern Introduction to Quantum Field Theory.* Oxford University Press, 2005
- R. A. Bertlmann, *Anomalies in Quantum Field Theory.* Oxford University Press, 1996.
- S. Pokorski, *Gauge Field Theories.* Cambridge University Press, 1987
- P. H. Frampton, *Gauge Field Theories.* Viley-VCH, 2008.
- V. P. Nair, *Quantum Field Theory: A Modern Perspective.* Springer, 2005.
- S. Scherer, *Introduction to Chiral Perturbation Theory*, Adv. Nucl. Phys. 27 (2003) 277, arXiv:hep-ph/0210398

Hadrons produce poles:

- S. Weinberg, *The Quantum Theory of Fields. Vol. I.* Cambridge University Press, 1995/96 (Sec. 10.2)
- M. D. Schwartz, *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2014 (Sec. 24.3)

Lattice QCD calculations of hadron masses:

- Z. Fodor, C. Hoelbling, *Light Hadron Masses from Lattice QCD*, Rev. Mod. Phys. 84 (2012) 449, arXiv:1203.4789 [hep-lat]
- R. A. Briceno, J. J. Dudek, R. D. Young, *Scattering processes and resonances from lattice QCD*, Rev. Mod. Phys. 90 (2018) 2, 025001 arXiv:1706.06223 [hep-lat]

Bethe-Salpeter equations:

- P. Maris, C. D. Roberts, *π and K meson Bethe-Salpeter amplitudes*, Phys. Rev. C 56 (1997) 3369, arXiv:nucl-th/9708029
- G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, C. S. Fischer, *Baryons as relativistic three-quark bound states*, Prog. Part. Nucl. Phys. 91 (2016) 1, arXiv:1606.09602 [hep-ph]

Flavor wave functions:

- H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, 1999
- A. Hosaka and H. Toki, *Quarks, Baryons and Chiral Symmetry.* World Scientific, 2001
- F. E. Close, *An Introduction to Quarks and Partons.* Academic Press, 1979
- T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics.* Oxford University Press, 1984
- F. Halzen and A. D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics.* Wiley, 1984.

Reviews on light exotic mesons:

- J. Pelaez, *From controversy to precision on the sigma meson: a review on the status of the non-ordinary $f_0(500)$ resonance*, Phys. Rept. 658, 1 (2016), arXiv:1510.00653 [hep-ph].
- C. A. Meyer, E. S. Swanson, *Hybrid Mesons*, Prog. Part. Nucl. Phys. 82 (2015), 21 arXiv:1502.07276 [hep-ph]
- W. Ochs, *The Status of Glueballs*, J. Phys. G 40 (2013) 043001, arXiv:1301.5183 [hep-ph]

Quark model classification of baryons:

- A. Hosaka and H. Toki, *Quarks, Baryons and Chiral Symmetry.* World Scientific, 2001
- E. Klempt and B. C. Metsch, *Multiplet classification of light-quark baryons.* Eur. Phys. J. A (2012) **48**,
- PDG review on the quark model, P.A. Zyla et al., Prog. Theor. Exp. Phys. 2020, 083C01 (2020)

Reviews on light baryons:

- E. Klempt, J.-M. Richard, *Baryon spectroscopy*, Rev. Mod. Phys. 82 2010 (1095), `arXiv:0901.2055 [hep-ph]`
- V. Crede, W. Roberts, *Progress Toward Understanding Baryon Resonances*, Reports on Progress in Physics 76 (2013) 7, `arXiv:1302.7299 [nucl-ex]`

Reviews on heavy exotics:

- H.-X. Chen, W. Chen, X. Liu, S.-L. Zhu, *The hidden-charm pentaquark and tetraquark states*, Phys. Rept. 639, 1-121 (2016), `arXiv:1601.02092 [hep-ph]`
- R. F. Lebed, R. E. Mitchell, E. S. Swanson, *Heavy-Quark QCD Exotica*, Prog. Part. Nucl. Phys. 93, 143 (2017), `arXiv:1610.04528 [hep-ph]`
- A. Esposito, A. Pilloni, A. D. Polosa, *Multiquark Resonances*, Phys. Rept. 668 (2017) 1, `arXiv:1611.07920 [hep-ph]`
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90, 015004 (2018), `arXiv:1705.00141 [hep-ph]`
- A. Ali, J. S. Lange, S. Stone, *Exotics: Heavy Pentaquarks and Tetraquarks*, Prog. Part. Nucl. Phys. 97 (2017) 123, `arXiv:1706.00610 [hep-ph]`
- S. L. Olsen, T. Skwarnicki, D. Zieminska, *Non-Standard Heavy Mesons and Baryons, an Experimental Review*, `arXiv:1708.04012 [hep-ph]`
- Y. R. Liu, H.-X. Chen, W. Chen, X. Liu, S.-L. Zhu, *Pentaquark and Tetraquark states*, Prog. Part. Nucl. Phys. 107 (2019) 237-320, `arXiv:1903.11976 [hep-ph]`
- N. Brambilla et al., *The XYZ states: experimental and theoretical status and perspectives*, Physics Reports 873 (2020) 1-154, `arXiv:1907.07583 [hep-ex]`